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ROYAL ASTRONOMICAL SOCIETY.

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SIR DAVID GILL, K.C.B., F.R.S., PRESIDENT, in the Chair.

Thomas William Bush, Lullenden Farm, East Grinstead, Surrey;
John Walton Capstick, M.A., D.Sc., Trinity College, Cambridge;
William Bennett Barton Freeland, J.P., Cathedral Registry Office, Chichester, Sussex;
R. W. Giblin, Director of the Royal Survey Department, Bangkok, Siam;
J. W. Short, The Observatory, Sydney, New South Wales, Australia; and
Lieut. William H. Tapp, The Hill, Bromley, Kent.

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

Professor A. W. Bickerton, Wainai Park, Christchurch, New Zealand (proposed by C. J. Merfield);
John Knight Fotheringham, M.A., D.Litt., Lecturer in Ancient History at King's College, London (proposed by H. H. Turner);
Edward Rupert Hicks, Gentleman, 67 Holland Road, Kensington, W. (proposed by W. E. Rolston);
Alfred Harrison Joy, M.A., Director of the Observatory, Syrian Protestant College, Beirût, Syria (proposed by H. H. Turner);
Rev. D. Kennedy, Meeanee, Napier, New Zealand (proposed by Sir David Gill);

William J. MacDonnell, Falmouth Chambers 4, Pitt Street 117, Sydney, New South Wales, Australia (proposed by C. J. Merfield); and

Stanley Maxwell, M.A., LL.B., Manor House School, Clapham Common (proposed by George Bruford).

Mr. W. E. Rolston, Mr. W. Shackleton and Dr. D. Smart were appointed Auditors of the Treasurer's accounts for the year 1909.

Sixty-four presents were announced as having been received since the last meeting, including, amongst others:—

Greenwich Observations, 1907; Groningen Publications, No. 23 (Kapteyn and de Sitter, Parallax of Hyades); Hamburg Observatory, Abhandlungen, vol. 1; presented by the respective observatories.

Astrographic Chart; 16 charts, presented by the Royal Observatory, Greenwich.

Photographs of solar prominences, portions of the Moon, and the planet Mars (enlarged transparencies), presented by the Mount Wilson Observatory; Eclipse of 1905 (engraving from drawings), presented by Dr. Drewett.

On certain Families of Periodic Orbits. By Sir George Darwin, K.C.B., F.R.S.

Introduction.

In my paper on "Periodic Orbits" in the restricted problem of the three bodies (*Acta Mathematica*, vol. xxi., 1897, p. 101) I attempted to make a classification of those orbits which are simply periodic, that is to say, re-entrant after a single circuit either round the Sun, or round the perturbing planet called Jove, or round one of the points of zero force. In that paper I only proposed to consider orbits in which the revolution is in the positive or "direct" sense, and the retrograde orbits were left for future consideration.

Amongst the orbits discussed there was a certain family of satellites which I called A. There were also certain other orbits resembling the figure 8, in which one loop was described round Jove and the second round that point of zero force which lies between the Sun and Jove. The results seemed to indicate that the orbits of these satellites, conjointly with the orbit of a satellite (a) oscillating about the point of zero force, and the figure-of-8 orbit belonged to the same family. But shortly after the publication of the paper it was pointed out both by Mr. S. S. Hough and by M. Poincaré that this kind of continuity was not possible, so that

the A orbits and the figure-of-8 orbits must in reality belong to different families; and indeed I had independently come to the same conclusion myself.

In consequence of this oversight my investigation left several points unexplained, for there was no indication as to the course of development of the A orbits after I had supposed them to cease to exist, and it remained to discover how they came to disappear from amongst the directly moving satellites. The origin of the figure-of-8 orbits also remained obscure.

To meet these deficiencies Mr. Hough wrote a paper entitled "On certain discontinuities connected with Periodic Orbits" (*Acta Mathematica*, vol. xxiv., 1901, p. 257). He there suggested, with great ability, the manner in which my classification should be completed, and indicated the existence of another family of figure-of-8 orbits which I had not detected.

My own experience has shown how hard it is, without actual computation, to attain full confidence as to the changes which orbits will undergo when the constant of relative energy changes. It was therefore desirable to test the accuracy of even well-reasoned conjectures, such as those of Mr. Hough.

In the present paper I shall show the results of my computations made with this object, and shall thereby confirm his conclusions in almost every detail. Indeed, he only went wrong, and that not seriously, where he relied implicitly on the completeness of my own work. I had not then considered what M. Burrau calls "orbits of ejection," which furnish the transitional forms between direct and retrograde orbits.* I had computed an orbit for $C = 39$ which started normally to the line of syzygy at a distance of only ·001 of the adopted unit of length, and not unnaturally it did not occur to me to suspect that if the initial pericentral distance were diminished yet further, the orbit would assume a very different form. Yet the computation of the orbit of ejection, in which the pericentral distance is reduced to zero, shows that such is the case. This oversight led to the erroneous conclusion (on which Mr. Hough relied) that the A orbit does not exist when $C = 39$. Mr. Hough supposed that it had already become retrograde when C had fallen to 39, whereas it will appear below that it only passes through the stage of ejection, on the way to becoming retrograde, when C has fallen to about 38·85.

The complete discussion of a family of orbits, many of whose members are direct, demands the consideration of retrograde orbits, because it would seem that sooner or later every family will pass through the ejectional stage. Moreover, superior planets, which are really revolving in a positive direction in space, will appear as retrograde when the motion is referred to rotating axes.

Those who have read my previous paper on this subject will be aware that the determination of the stability of a periodic orbit presents an arduous task. Mr. Hough has suggested to me a mode

* See *Astronomische Nachrichten*, No. 3230, vol. 135, and No. 3251, vol. 136 (1894).

of procedure which differs completely from that which I had adopted; it is explained below, but in my own words. Whatever plan is followed, the computation must necessarily remain laborious, and for orbits with sharp flexures both methods become almost impracticable. In such cases Mr. Hough's method appears to be somewhat less intractable than that of the infinite determinant, although hardly less laborious. On the whole, I think his plan is somewhat preferable to mine, although for orbits of fairly continuous changes of curvature it does not seem to possess any conspicuous advantage. I apply his method to the discussion of several of the orbits determined in this paper.

The various subjects considered here are so far connected together that they all aim at the completion of our knowledge of periodic orbits, but they do not form the matter of a single continuous argument, and therefore the paper is divided into several parts.

I begin in Part I. with the investigation of Mr. Hough's method of determining the stability of an orbit, and reduce the formulæ to convenient forms for computation.

There follows next in Part II. the discussion of certain retrograde orbits. It had originally been my intention to obtain a complete knowledge of these orbits, but as I see no immediate prospect of continuing these laborious investigations, I set forth all the results obtained up to the present time. As regards the satellites, with which Mr. Hough was concerned, the results are entirely confirmatory of his suggestions. Several members of the family of superior planets (only apparently retrograde) are also shown, and incidentally we obtain some insight into other families of orbits which must exist.

It has been already remarked that orbits of ejection are transitional forms between direct and retrograde orbits, and they therefore possess, as M. Burrau perceived, a special degree of importance. Accordingly, in Part III. analytical formulæ are found which enable us to trace these orbits for a considerable distance from the point of ejection. A large number of such orbits, completed by quadratures, are presented. I also add some results as to the form of the cusp when the body reaches the curve of zero velocity.

Part IV. is devoted to the subject which affords the principal reason for this paper, namely, the completion of the deficiencies in my previous work.

Since the figure-of-8 orbits were erroneously classified along with the satellites A, I now denote them by A′. The new family of figures-of-8, foreseen by Mr. Hough, is also determined and denoted A″. Figures of these two families are drawn, and I also show the fate of the satellites A, and how they become retrograde in passing through the ejectional form. An attempt is made to discriminate between the orbits A′ and A″ near the point of their coalescence, and it is shown that it is probably A″ which is stable.

Finally, Part V. contains a discussion of the light which is

thrown on the development of several of the families of orbits by means of these investigations, and I indicate various new families of orbits which must spring into existence.

This paper will ultimately be reproduced in the fourth volume of my collected papers, and I propose to give there in an appendix the numerical values from which many of the figures were drawn. I shall thus be following the course pursued in my original paper, but it would make this paper somewhat too long for its present place of publication to give these numerical details here also.

I. The Stability of Orbits.

§ 1. *Determination of the Stability of a Periodic Orbit.*

In my paper on "Periodic Orbits" the criterion of stability of the orbits was found by means of Hill's infinite determinant, but the method virtually fails when the orbit has sharp flexures. A different method of treatment will therefore now be explained.

In § 10 of that paper I put $\sigma = \pi s/S$, where s is the arc of orbit measured from the initial syzygy, and S is the complete arc of the periodic orbit. The equation from which the stability was determined was

$$\frac{d^2\delta q}{d\sigma^2} + \Phi\delta q = 0 ,$$

where Φ is an even function of σ completing its period when σ increases from 0 to π.

The following investigation, as far as regards the analytical principles involved, was given to me by Mr. S. S. Hough, although I am responsible for the manner in which it is presented.

The differential equation remains unchanged when σ becomes equal to $\sigma + \pi$. Hence, if $\delta q = F(\sigma)$ is a solution, $F(\sigma + \pi)$ is the same solution, only differing from the first by a constant factor.

Therefore if ν is a constant

$$F(\sigma + \pi) = \nu F(\sigma) .$$

Let $\phi(\sigma)$ be a solution, subject to the condition that when $\sigma = 0$, $\delta q = 1$, $d\delta q/d\sigma = 0$. That is to say, $\phi(0) = 1$, $\phi'(0) = 0$.

Let $\psi(\sigma)$ be another solution, subject to the condition that when $\sigma = 0$, $\delta q = 0$, $d\delta q/d\sigma = 1$. That is to say, $\psi(0) = 0$, $\psi'(0) = 1$.

It is clear that the general solution of the equation is

$$\delta q = F(\sigma) = A\phi(\sigma) + B\psi(\sigma) ,$$

where A, B are arbitrary constants.

Since $\phi(\sigma + \pi)$, $\psi(\sigma + \pi)$ are solutions of the equation, we must have

$$\left.\begin{aligned} \phi(\sigma + \pi) &= \alpha\phi(\sigma) + \beta\psi(\sigma) , \\ \psi(\sigma + \pi) &= \gamma\phi(\sigma) + \delta\psi(\sigma) , \end{aligned}\right\} \quad . \quad . \quad (1)$$

where $\alpha, \beta, \gamma, \delta$ are the appropriate values of the arbitrary constants A, B to correspond with these special cases.

If in the equation $F(\sigma+\pi) = \nu F(\sigma)$ we substitute for I' its value in terms of ϕ and ψ, we obtain

$$A[\phi(\sigma+\pi) - \nu\phi(\sigma)] + B[\psi(\sigma+\pi) - \nu\psi(\sigma)] = 0 .$$

Substituting further for $\phi(\sigma+\pi)$, $\psi(\sigma+\pi)$ their values, we obtain

$$[A(\alpha-\nu) + B\gamma]\phi(\sigma) + [A\beta + B(\delta-\nu)]\psi(\sigma) = 0 .$$

Since this must be satisfied for all values of σ, we have

$$A(\alpha-\nu) + B\gamma = 0 ,$$

$$A\beta + B(\delta-\nu) = 0 .$$

Whence, on eliminating A and B,

$$\nu^2 - \nu(\alpha+\delta) + (\alpha\delta - \beta\gamma) = 0 .$$

It will now be proved that the last term in this quadratic is unity.

Since ϕ and ψ are solutions of the differential equation,

$$\frac{d^2\phi}{d\sigma^2} + \Phi\phi = 0 , \quad \frac{d^2\psi}{d\sigma^2} + \Phi\psi = 0 .$$

Multiplying the first of these by ψ, subtracting it from the second when multiplied by ϕ, and integrating, we find

$$\phi\psi' - \phi'\psi = \text{constant.}$$

Putting $\sigma = 0$, we see that the constant is unity.

But (1) with $\sigma = 0$, shows that

$$\phi(\pi) = \alpha , \quad \phi'(\pi) = \beta ,$$

$$\psi(\pi) = \gamma , \quad \psi'(\pi) = \delta .$$

Therefore $\alpha\delta - \beta\gamma = 1$.

The equation for ν may therefore be written in the form

$$\tfrac{1}{2}\left(\nu + \frac{1}{\nu}\right) = \tfrac{1}{2}(\alpha+\delta) .$$

Since ϕ and ψ are respectively odd and even functions of σ,

$$\phi(-\sigma) = \phi(\sigma) , \qquad \psi(-\sigma) = -\psi(\sigma) ,$$

$$\phi'(-\sigma) = -\phi'(\sigma) , \quad \psi'(-\sigma) = \psi'(\sigma) .$$

On putting $\sigma = -\tfrac{1}{2}\pi$ in (1) and in the pair of equations derived therefrom by differentiation, we obtain

$$\phi(\tfrac{1}{2}\pi) = \alpha\phi(\tfrac{1}{2}\pi) - \beta\psi(\tfrac{1}{2}\pi) , \quad \phi'(\tfrac{1}{2}\pi) = -\alpha\phi'(\tfrac{1}{2}\pi) + \beta\psi'(\tfrac{1}{2}\pi) ,$$

$$\psi(\tfrac{1}{2}\pi) = \gamma\phi(\tfrac{1}{2}\pi) - \delta\psi(\tfrac{1}{2}\pi) , \quad \psi'(\tfrac{1}{2}\pi) = -\gamma\phi'(\tfrac{1}{2}\pi) + \delta\psi'(\tfrac{1}{2}\pi) .$$

Whence

$$\frac{\phi(\frac{1}{2}\pi)}{\psi(\frac{1}{2}\pi)} = \frac{\beta}{\alpha - 1} = \frac{\delta + 1}{\gamma}, \quad \frac{\psi'(\frac{1}{2}\pi)}{\phi'(\frac{1}{2}\pi)} = \frac{\alpha + 1}{\beta} = \frac{\gamma}{\delta - 1}.$$

Therefore

$$\frac{\alpha + 1}{\alpha - 1} = \frac{\delta - 1}{\delta + 1} = \frac{\phi(\frac{1}{2}\pi)\psi'(\frac{1}{2}\pi)}{\phi'(\frac{1}{2}\pi)\psi(\frac{1}{2}\pi)}.$$

Since $\phi(\frac{1}{2}\pi)\psi'(\frac{1}{2}\pi) - \phi'(\frac{1}{2}\pi)\psi(\frac{1}{2}\pi) = 1$, we have

$$\alpha = \delta = \tfrac{1}{2}(\alpha + \delta) = \phi(\tfrac{1}{2}\pi)\psi'(\tfrac{1}{2}\pi) + \phi'(\tfrac{1}{2}\pi)\psi(\tfrac{1}{2}\pi).$$

We may therefore write the equation for ν in any one of the five following forms:—

$$\begin{aligned}\tfrac{1}{2}\left(\nu + \frac{1}{\nu}\right) &= \phi(\pi) = \psi'(\pi) = \phi(\tfrac{1}{2}\pi)\psi'(\tfrac{1}{2}\pi) + \phi'(\tfrac{1}{2}\pi)\psi(\tfrac{1}{2}\pi) \\ &= 1 + 2\phi'(\tfrac{1}{2}\pi)\psi(\tfrac{1}{2}\pi) = 2\phi(\tfrac{1}{2}\pi)\psi'(\tfrac{1}{2}\pi) - 1.\end{aligned}$$

It remains to determine the meaning of ν in terms of c, Hill's constant, or more exactly in my notation half of Hill's constant.

In § 10, p. 142, of "Periodic Orbits" the solution of the differential equation was shown to be

$$\delta q = \Sigma_j[(b_j + e_j)\cos(c + 2j)\sigma + (b_j - e_j)\sqrt{-1}\sin(c + 2j)\sigma],$$

where Σ_j denotes summation for all integral values of j from $+\infty$ to $-\infty$.

In the solution ϕ, $\phi(0) = 1$, $\phi'(0) = 0$. Hence in this case $e_{-j} = b_j$, and

$$\phi(\sigma) = 2\Sigma_j b_j \cos(c + 2j)\sigma.$$

With $\sigma = 0$, we see that $2\Sigma_j b_j = 1$, $\phi'(0) = 0$. It follows that

$$\phi(\pi) = 2\cos\pi c\,\Sigma b_j = \cos\pi c.$$

In the same way, on choosing the constants to produce the other solution ψ, we find

$$\psi'(\pi) = \cos\pi c.$$

It follows that $\frac{1}{2}\left(\nu + \frac{1}{\nu}\right) = \cos\pi c$, and our solutions may be written

$$\begin{aligned}\cos\pi c &= \phi(\pi) = \psi'(\pi), \\ \cos^2\tfrac{1}{2}\pi c &= \phi(\tfrac{1}{2}\pi)\psi'(\tfrac{1}{2}\pi), \\ \sin^2\tfrac{1}{2}\pi c &= -\phi'(\tfrac{1}{2}\pi)\psi(\tfrac{1}{2}\pi).\end{aligned}$$

The last of these has the form used in "Periodic Orbits," where the result was

$$\sin^2\tfrac{1}{2}\pi c = \Delta\sin^2\tfrac{1}{2}\pi\sqrt{\Phi_0},$$

Δ being a certain infinite determinant.

If c is any solution of the equation, all others are included in $\pm c \pm 2j$, and I choose the fundamental value of c to be that which

lies nearest to $\sqrt{\Phi_0}$, or the positive square root of the mean value of Φ.

The evaluation of c, and therefore the determination of stability, accordingly depends on the values of two special solutions, after half a circuit of the orbit has been performed when σ has increased from 0 to $\frac{1}{2}\pi$.

The periodic orbit having been determined by quadrature, the value of Φ may be found at a succession of points distributed along the orbit, and therefore it is a matter of quadrature to find the two required solutions.

We may start from the line of syzygies with two quadratures, denoted by suffixes 1 and 2; the first begins with $(\delta q_1)_0 = 1$, $\left(\frac{d\delta q_1}{d\sigma}\right)_0 = 0$, and the second with $(\delta q_2)_0 = 0$, $\left(\frac{d\delta q_2}{d\sigma}\right)_0 = 1$, where the suffix 0 indicates the initial value. If we denote the values of any quantity after passing half round the periodic orbit by the suffix s, we have

$$\sin^2 \tfrac{1}{2}\pi c = -(\delta q_2)_s \left(\frac{d\delta q_1}{d\sigma}\right)_s .$$

For practical work the form of this solution may be modified with advantage. The arc of orbit s is the independent variable which arises naturally, and the differential equation is (as shown in (39), p. 140 of "Periodic Orbits") then

$$\frac{d^2\delta q}{ds^2} + \Psi \delta q = 0 .$$

I subsequently, in § 10, changed the independent variable to $\sigma = \pi s/S$, where S is the complete arc of the periodic orbit. To effect this change I put $\Phi = S^2\Psi/\pi^2$. Hence with s as independent variable the second of the two solutions starts with $\left(\frac{d\delta q_2}{ds}\right)_0 = \frac{\pi}{S}$, and at every point the solutions satisfy the condition

$$\delta q_1 \frac{d\delta q_2}{ds} - \delta q_2 \frac{d\delta q_1}{ds} = \frac{\pi}{S} .$$

The equation for c then becomes

$$\sin^2 \tfrac{1}{2}\pi c = -\frac{S}{\pi}(\delta q_2)_s \left(\frac{d\delta q_1}{ds}\right)_s .$$

But for practical numerical work some further transformations will be found convenient.

In computing a periodic orbit I proceed step by step, the common difference of the arcs being Δs, the value of which may, however, change from time to time. In using the formulæ of finite differences it is convenient to make the common difference of the independent variable equal to unity. This will be secured by adopting z as independent variable where

$$dz = d\left(\frac{s - s_n}{\Delta s}\right),$$

s_n being the arc of the orbit measured from the initial syzygy up to the nth point in the orbit. If we make

$$\Omega = \Psi(\Delta s)^2,$$

the differential equation may be written

$$\frac{d^2\delta q}{dz^2} + \Omega\delta q = 0 .$$

Since Δs is usually a simple fraction such as ·01 or ·02, the multiplication of the Ψ's to form the Ω's is very simple.

Suppose that $\Omega^{(n-1)}$, $\Omega^{(n)}$, $\Omega^{(n+1)}$, $\Omega^{(n+2)}$ are four successive values of Ω; that ΔE^{-1}, Δ, ΔE are the three first differences; $\Delta^2 E^{-1}$, Δ^2 the two second differences; and $\Delta^3 E^{-1}$ the third difference. The first difference falling between the nth and $(n+1)$th entry in the table of Ω is Δ, the mean second difference is $\frac{1}{2}(\Delta + \Delta E^{-1})$, the third difference is $\Delta^3 E^{-1}$. If we call these three differences δ, δ^2, δ^3, Bessel's formula of interpolation is

$$\Omega^{(z)} = \Omega^{(n)} + z\delta + \tfrac{1}{2}z(z-1)\delta^2 + \tfrac{1}{6}z(z-1)(z-\tfrac{1}{2})\delta^3 .$$

If we write

$$\Omega_0 = \Omega^{(n)}, \quad \Omega_1 = \delta - \tfrac{1}{2}\delta^2 + \tfrac{1}{12}\delta^3, \quad \Omega_2 = \tfrac{1}{2}\delta^2 - \tfrac{1}{4}\delta^3, \quad \Omega_3 = \tfrac{1}{6}\delta^3,$$

we have

$$\Omega^{(z)} = \Omega_0 + z\Omega_1 + z^2\Omega_2 + z^3\Omega_3 + \ldots .$$

Dropping the (z) on the left as being unnecessary, we may therefore write

$$\Omega = \sum_0^\infty \Omega_i z^i .$$

If we assume

$$\delta q_1 = \sum_0^\infty p_i z^i ,$$

we have

$$\Omega\delta q_1 = \sum_{i=0}^{i=\infty} \sum_{j=0}^{j=i} \Omega_j p_{i-j} z^i ;$$

by integration of which we have from the differential equation,

$$\frac{d\delta q_1}{dz} = p_1 - \sum_{i=0}^{i=\infty} \sum_{j=0}^{j=i} \frac{1}{i+1} \Omega_j p_{i-j} z^{i+1},$$

$$\delta q_1 = p_0 + p_1 z - \sum_{i=0}^{i=\infty} \sum_{j=0}^{j=i} \frac{1}{(i+1)(i+2)} \Omega_j p_{i-j} z^{i+2} .$$

By equating the coefficients of the assumed form of δq_1 with this result, we find

$$p_i = -\frac{1}{i(i+1)}\left[\Omega_0 p_{i-1} + \Omega_1 p_{i-2} + \ldots . + \Omega_{i-2} p_1 + \Omega_{i-1} p_0\right] .$$

Let there be a second solution, and let it be expressed in the same form, save that q's replace p's.

The first solution starts with

$$(\delta q_1)_0 = 1\,, \quad \left(\frac{d\delta q_1}{dz}\right)_0 = 0\,,$$

and the second with

$$(\delta q_2)_0 = 0\,, \quad \left(\frac{d\delta q_2}{dz}\right)_0 = \frac{\pi\Delta s}{S}\,.$$

At every point of the orbit we have

$$\delta q_1 \frac{d\delta q_2}{dz} - \delta q_2 \frac{d\delta q_1}{dz} = \frac{\pi\Delta s}{S}\,.$$

As this is true at the point $z=0$, which is the nth point in the periodic orbit,

$$p_0 q_1 - p_1 q_0 = \frac{\pi\Delta s}{S}\,.$$

If we desire to record in the notation the meanings of p_0, p_1, q_0, q_1, this may be written

$$\left(\delta q_1 \frac{d\delta q_2}{dz} - \delta q_2 \frac{d\delta q_1}{dz}\right)_n = \frac{\pi\Delta s}{S}\,.$$

This is the condition that at the nth point the solutions shall correspond to the supposed initial conditions. It will appear that if it holds good at the nth point, it does so also at the $(n+1)$th.

The value $z=1$ corresponds to the passage from the nth to the $(n+1)$th point. Therefore

$$(\delta q_1)_{n+1} = \sum_0^\infty p_i\,, \qquad (\delta q_2)_{n+1} = \sum_0^\infty q_i\,,$$

$$\left(\frac{d\delta q_1}{dz}\right)_{n+1} = \sum_0^\infty i p_i\,, \quad \left(\frac{d\delta q_2}{dz}\right)_{n+1} = \sum_0^\infty i q_i\,.$$

It has been supposed that four computed values of Ω at the points $n-1$, n, $n+1$, $n+2$ give a sufficient representation of Ω throughout the range n to $n+1$. These four values afford the means of computing Ω_0, Ω_1, Ω_2, Ω_3 as given above. It will be supposed that the higher Ω's are negligible, and that it is unnecessary to consider their squares and products, excepting Ω_0^2 and $\Omega_0\Omega_1$.

The solution of the successive equations for the p's gives

$$p_2 = -\tfrac{1}{1.2}\Omega_0 p_0\,, \quad p_3 = -\tfrac{1}{2.3}(\Omega_0 p_1 + \Omega_1 p_0)\,,$$

$$p_4 = -\tfrac{1}{3.4}\left[(\Omega_2 - \tfrac{1}{2}\Omega_0^2)p_0 + \Omega_1 p_1\right],$$

$$p_5 = -\tfrac{1}{4.5}\left[(\Omega_3 - \tfrac{2}{3}\Omega_0\Omega_1)p_0 + (\Omega_2 - \tfrac{1}{6}\Omega_0^2)p_1\right],$$

$$p_6 = -\tfrac{1}{5.6}(\Omega_3 - \tfrac{1}{4}\Omega_0\Omega_1)p_1\,.$$

To the degree of approximation adopted all higher p's are to be neglected, although it would be easy to give them to any desired degree of accuracy.

Substituting these values of the p's in the foregoing expression, and noting that $(\delta q_1)_n = p_0$, $\left(\frac{d\delta q_1}{dz}\right)_n = p_1$, we find

$$(\delta q_1)_{n+1} = A_0(\delta q_1)_n + A_1\left(\frac{d\delta q_1}{dz}\right)_n,$$

$$\left(\frac{d\delta q_1}{dz}\right)_{n+1} = -B_0(\delta q_1)_n + B_1\left(\frac{d\delta q_1}{dz}\right)_n.$$

Similar equations hold good with suffix **2**, and the same A_0, A_1, B_0, B_1.

The values of these coefficients are found to be

$$A_0 = 1 - \tfrac{1}{1.2}\Omega_0 - \tfrac{1}{2.3}\Omega_1 - \tfrac{1}{3.4}\Omega_2 - \tfrac{1}{4.5}\Omega_3 + \tfrac{1}{24}\Omega_0^2 + \tfrac{1}{30}\Omega_0\Omega_1,$$

$$A_1 = 1 - \tfrac{1}{2.3}\Omega_0 - \tfrac{1}{3.4}\Omega_1 - \tfrac{1}{4.5}\Omega_2 - \tfrac{1}{5.6}\Omega_3 + \tfrac{1}{120}\Omega_0^2 + \tfrac{1}{120}\Omega_0\Omega_1,$$

$$B_0 = \Omega_0 + \tfrac{1}{2}\Omega_1 + \tfrac{1}{3}\Omega_2 + \tfrac{1}{4}\Omega_3 - \tfrac{1}{6}\Omega_0^2 - \tfrac{1}{6}\Omega_0\Omega_1,$$

$$B_1 = 1 - \tfrac{1}{2}\Omega_0 - \tfrac{1}{3}\Omega_1 - \tfrac{1}{4}\Omega_2 - \tfrac{1}{5}\Omega_3 + \tfrac{1}{24}\Omega_0^2 + \tfrac{1}{20}\Omega_0\Omega_1.$$

The form of the solution shows that

$$\left(\delta q_1\frac{d\delta q_2}{dz} - \delta q_2\frac{d\delta q_1}{dz}\right)_{n+1} = (A_0B_1 + A_1B_0)\left(\delta q_1\frac{d\delta q_2}{dz} - \delta q_2\frac{d\delta q_1}{dz}\right)_n.$$

It is easy to show that, to the degree of approximation adopted,

$$A_0B_1 + A_1B_0 = 1.$$

Hence, if the condition is satisfied at the nth point, it is so also at the $(n+1)$th.

Accordingly, the integration can be effected by computing the Ω's at each point, combining them to form A_0, A_1, B_0, B_1, and thence computing δq_1, δq_2, $d\delta q_1/dz$, $d\delta q_2/dz$.

If a suffix s attached to any symbol indicates its value after carrying the integration half round the periodic orbit, we have

$$\sin^2\tfrac{1}{2}\pi c = -\frac{S}{\pi\Delta s}(\delta q_2)_s\left(\frac{d\delta q_1}{dz}\right)_s.$$

If this quantity is positive and less than unity, the orbit is stable; if negative or greater than unity, it is unstable.

The computation is somewhat abridged if we do not actually compute Ω_2, Ω_3, for the results may be expressed in terms of the δ's, which are easily computed from a difference table of the Ω_0's.

I find that

$$A_0 = 1 - \tfrac{1}{2}[(\Omega_0 - \tfrac{1}{12}\delta^2) + \tfrac{1}{3}\delta + \tfrac{1}{360}\delta^3] + \tfrac{1}{24}\Omega_0^2 + \tfrac{1}{30}\Omega_0\Omega_1 ,$$

$$B_0 = (\Omega_0 - \tfrac{1}{12}\delta^2) + \tfrac{1}{2}\delta - \tfrac{1}{6}\Omega_0^2 - \tfrac{1}{6}\Omega_0\Omega_1 ,$$

$$B_1 = A_0 - \tfrac{1}{6}\delta + \tfrac{1}{360}\delta^3 + \tfrac{1}{60}\Omega_0\Omega_1 ,$$

$$A_1 = 1 - \tfrac{1}{6}(B_0 - \tfrac{1}{60}\delta^2) - \tfrac{7}{360}\Omega_0^2 - \tfrac{7}{360}\Omega_0\Omega_1 .$$

It is well to verify the relationship $A_0B_1 + A_1B_0 = 1$, and those between δq_1, δq_2 and their differentials at each stage of the computation.

In order to test this method of determining stability, I applied it to a case in which the stability seemed to be well determined by means of the infinite determinant. The orbit chosen was the second of those for which tabular values are given in my previous paper, viz. $C = 40{\cdot}25$, $x_0 = 1{\cdot}1150$. The determinantal method gave $\sin^2 \tfrac{1}{2}\pi c = {\cdot}0630$, and $c = 2{\cdot}161$. This new method gives ·0625 and 2·157 respectively; thus the agreement is excellent.*

I have also applied the method to other orbits and to one of the figure-of-8 orbits, as will be explained hereafter.

It will now be shown that the values of $(\delta q_2)_s$ and $\left(\frac{d\delta q_1}{dz}\right)_s$ may be obtained otherwise than by the method of quadrature already explained. It will be proved that the computation of two orbits, having certain relationships to the periodic orbit, suffices for this end.

If our two solutions had corresponded with $(\delta q_1)_0 = \eta$, $\left(\frac{d\delta q_1}{dz}\right)_0 = 0$, and $(\delta q_2)_0 = 0$, $\left(\frac{d\delta q_2}{dz}\right)_0 = \eta'$, the former work, in which we took $\eta = 1$ and $\eta' = \frac{\pi\Delta s}{S}$, would have applied again with very slight changes. It is easy indeed to see that the solution becomes

$$\sin^2 \tfrac{1}{2}\pi c = \frac{1}{\eta\eta'} \cdot \frac{S}{\pi\Delta s}\left(\delta q_2 \frac{d\delta q_1}{dz}\right)_s .$$

When the suffix s is attached to any symbol, or group of symbols, it is meant to denote the value after half the circuit of the periodic orbit has been performed, starting from the initial syzygy. The suffix 0 is to denote the corresponding value at that initial syzygy.

In the paper on "Periodic Orbits" (§ 8, p. 133), δp, δs denoted the displacements whereby we pass from an orbit of reference to the corresponding point in a varied orbit. The displacement along the outward normal is δp, that along the tangent is δs. The symbol $\delta\phi$ means the corresponding variation of ϕ.

* I shall give in my Vol. IV. a table showing the march of δq_1, δq_2 and of their differential coefficients, and exhibiting the further steps in the computation.

In (41) and (42) of that paper it is shown that

$$\frac{d}{ds}\left(\frac{\delta s}{V}\right) = -\frac{2\delta q}{V^{\frac{3}{2}}}\left(\frac{1}{R}+\frac{n}{V}\right),$$

$$\delta\phi = -\frac{1}{V^{\frac{3}{2}}}\left[\frac{d\delta q}{ds} - \tfrac{1}{2}\delta q\left(\frac{dV}{Vds}\right)\right] + \frac{\delta s}{R},$$

also

$$\delta p = \frac{\delta q}{V^{\frac{1}{2}}}.$$

The first of these three will not be needed, but it is given for the sake of completeness.

In the first solution we are supposed to start with

$$(\delta q_1)_0 = \eta, \quad \left(\frac{d\delta q_1}{dz}\right)_0 = \left(\frac{d\delta q_1}{ds}\right)_0 = 0.$$

Suppose that at the initial syzygy $(\delta s_1)_0 = 0$, $(\delta p_1)_0 = \epsilon$ (say ·001 as a convenient variation). Then we are proposing to trace an orbit starting from syzygy at distance from the sun S greater by ϵ than the initial starting-point x_0 of the periodic orbit. Thus we may say that $(x_1)_0 = x_0 + \epsilon$. It remains to consider the angle at which we shall start this new orbit. At syzygies V is a maximum or minimum, so that $dV/ds = 0$; also by hypothesis $(\delta s_1)_0 = 0$. Hence our second equation becomes

$$(\delta\phi_1)_0 = -\left(\frac{1}{V^{\frac{3}{2}}}\frac{d\delta q}{ds}\right)_0.$$

If therefore $\left(\frac{d\delta q}{ds}\right)_0 = 0$, we shall have $(\delta\phi_1)_0 = 0$.

In our proposed first orbit this condition is fulfilled, and therefore $(\delta\phi_1)_0 = 0$. That is to say, we are to start at right angles to the line of syzygy as in the periodic orbit, but at a slightly greater distance from S.

The definition of δq shows us that

$$(\delta q_1)_0 = (V^{\frac{1}{2}}\delta p_1)_0.$$

Hence

$$\eta = V_0^{\frac{1}{2}}\epsilon.$$

It follows that in tracing the proposed new orbit we are evaluating the first solution.

When we have traversed half the orbit $(\delta s_1)_s$ will not be zero, so that the corresponding point in the varied orbit will not be on the line of syzygies. But supposing the varied orbit to differ but slightly from the periodic orbit, we may denote the arc in the varied orbit from the starting-point to the second syzygy by $\frac{1}{2}S_1$. Then it is clear that $(\delta s_1)_s$ must be very nearly equal to $\frac{1}{2}S_1 - \frac{1}{2}S$, and we have merely to deduct the half arc of the periodic orbit from the half arc of the varied orbit in order to find it.

At this second syzygy V is again stationary, and the equation for $\delta\phi$ becomes

$$(\delta\phi_1)_s = -\left(\frac{1}{V^{\frac{3}{2}}}\frac{d\delta q_1}{ds}\right)_s + \left(\frac{\delta s_1}{R}\right)_s$$

$$= -\frac{1}{\Delta s}\left(\frac{1}{V^{\frac{3}{2}}}\frac{d\delta q_1}{dz}\right)_s + \left(\frac{\delta s_1}{R}\right)_s.$$

But $(\delta\phi_1)_s$ is the excess of the final ϕ in the varied orbit above its value in the periodic orbit, and therefore can be determined from the orbit as traced.

Hence we have

$$\left(\frac{d\delta q_1}{dz}\right)_s = V^{\frac{3}{2}}\Delta s\left[\frac{\delta s_1}{R} - \delta\phi_1\right]_s.$$

If x_s is the value of the abscissa x at the second crossing of the line of syzygy in the periodic orbit, $x_s + (\delta x_1)_s$ may be taken to represent the abscissa at this second crossing in the varied orbit. It is clear that if the varied orbit differs but little from the periodic, and therefore cuts the syzygies nearly at right angles, and if x_0 (or for satellites $x_0 - 1$) is positive, $(\delta p_1)_s$ will be equal to $-(\delta x_1)_s$. In such a case as the figure-of-8 orbits A′ considered hereafter, however, $(\delta p_1)_s$ will be equal to $+(\delta x_1)_s$, but we may consider that case by a mere change of sign.

Hence we have

$$(\delta q_1)_s = -(V^{\frac{1}{2}}\delta x_1)_s.$$

It follows that the tracing of this varied orbit affords the means of computing δq_1 and its differential coefficient after half a circuit has been performed.

Now, consider the second solution denoted by suffix 2. Suppose we start with $(\delta p_2)_0 = 0$, $(\delta s_2)_0 = 0$, but with $(\delta\phi_2)_0$ equal to $-\epsilon'$ (say for example $-1°$). Thus we start from the same point as for the periodic orbit, but not quite at right angles to the line of syzygy.

We have initially

$$(\delta\phi_2)_0 = -\frac{1}{\Delta s}\left(\frac{1}{V^{\frac{3}{2}}}\frac{d\delta q_2}{dz}\right)_0.$$

This may be written

$$\epsilon' = \frac{1}{\Delta s}\frac{1}{V_0^{\frac{1}{2}}}\eta'.$$

So that

$$\eta' = \Delta s V_0^{\frac{1}{2}}\epsilon'.$$

Having found η' in terms of ϵ' (say $1°$), the rest of the solution follows the same lines as the first, and therefore

$$(\delta q_2)_s = -(V^{\frac{1}{2}}\delta x_2)_s,$$

$$\left(\frac{d\delta q_2}{dz}\right)_s = V^{\frac{3}{2}}\Delta s\left(\frac{\delta s_2}{R} - \delta\phi_2\right)_s.$$

Thus the computation of these two additional orbits affords the means of determining the stability by means of the formula

$$\sin^2 \tfrac{1}{2}\pi c = -\frac{1}{\eta\eta'}\frac{S}{\pi\Delta s}\left(\delta q_2 \frac{d\delta q_1}{dz}\right)_s .$$

I think that this would be the most accurate method of determining stability in cases where there are sharp flexures, but I have not carried out the laborious computations in any case.

A variation of this procedure would result from the consideration of two orbits neither of which is exactly periodic, but I leave the reader to discuss this.

§ 2. *Interpretation of the Formulæ for the Motion of the Pericentre.*

In determining the stability of direct orbits, c was evaluated, and if it involved an imaginary quantity the orbit was unstable. If it was real, certain functions gave the motion of the pericentre. In determining these functions the notation used was as follows: T synodic period, n mean motion of Jove, $d\omega/dt$ mean motion of pericentre in space. The functions were:—

(1) The regression of pericentre in the synodic period referred to moving axes, or

$$T\left(n - \frac{d\omega}{dt}\right) = 2\pi(\tfrac{1}{2}c - 1) .$$

(2) The advance of pericentre in space in the synodic period, or

$$T\frac{d\omega}{dt} = nT - 2\pi(\tfrac{1}{2}c - 1) .$$

(3) The advance of pericentre in space in the sidereal period, or

$$\frac{2\pi T}{2\pi + nT}\frac{d\omega}{dt} = 2\pi\left(1 - \frac{\frac{1}{2}c}{1 + nT/2\pi}\right) .$$

When we consider orbits either really or apparently retrograde, some modification is needed in these functions.

It is well in all cases to adopt as the fundamental value of c that which lies nearest to $+\sqrt{\Phi_0}$ or $+\frac{S}{\pi}\sqrt{\Psi_0}$. As before, $\frac{1}{2}c$, regarded as positive, is the ratio of the synodic to the anomalistic period, although the corresponding angular velocities may have different signs.

Consider first the case of apparently retrograde orbits in which $2\pi/T$ is less than n. Here the orbit is direct in space. We must then change the sign of T in the above formulæ because the motion is apparently retrograde, and that of c because the two angular

velocities are in opposite senses. Hence we get

$$(1)\quad T\left(n-\frac{d\omega}{dt}\right) = 2\pi(\tfrac{1}{2}c+1),$$

$$(2)\quad T\frac{d\omega}{dt} = nT-2\pi(\tfrac{1}{2}c+1),$$

$$(3)\quad \frac{2\pi T}{2\pi-nT}\frac{d\omega}{dt} = 2\pi\left(1-\frac{\tfrac{1}{2}c}{nT/2\pi-1}\right).$$

If the orbit is really retrograde $2\pi/T$ is greater than n, and the two angular velocities are in the same sense; hence we only change the sign of T. We thus obtain

$$(1)\quad T\left(n-\frac{d\omega}{dt}\right) = -2\pi(\tfrac{1}{2}c-1),$$

$$(2)\quad T\frac{d\omega}{dt} = nT+2\pi(\tfrac{1}{2}c-1),$$

$$(3)\quad \frac{2\pi T}{2\pi-nT}\frac{d\omega}{dt} = -2\pi\left(1-\frac{\tfrac{1}{2}c}{1-nT/2\pi}\right).$$

A detailed consideration of these cases shows that these interpretations are correct. My attention was called to the point by the difficulty which arose when, on using the old formulæ, I seemed to find a very large movement of pericentre in orbits which were clearly very stable.

II. Retrograde Orbits.

§ 3. *Retrograde Satellites.*

In § 5 of his paper Mr. Hough drew conjecturally a number of retrograde satellites, and indicated the existence of a retrograde periodic orbit. My object was to test his conjectures, and with that view I chose the case of $C=40$, and here as elsewhere took the Sun's mass ν as ten times as great as that of Jove, which was unity. The results of the computations are exhibited in Fig. 1.

If the point of departure were at the curve of zero velocity on the side of J remote from S, which in Fig. 1 and in all the other figures is supposed to be to the left of J at a distance unity, the orbit would be cusped and the satellite would revolve about J in the positive direction. I have not drawn this cusped curve, but the first orbit shown is one in which the projection is retrograde at a distance $x_0=1{\cdot}23$; it is shown in chain-dot. It will be seen that as this satellite falls towards J it loses its retrograde character, so that it revolves round Jove in the positive direction. The next one, $x_0=1{\cdot}22$, has the same character, but the pericentral distance would be very small, and a further diminution of x_0 would give a satellite which would strike Jove. When $x_0=1{\cdot}20$ we have a

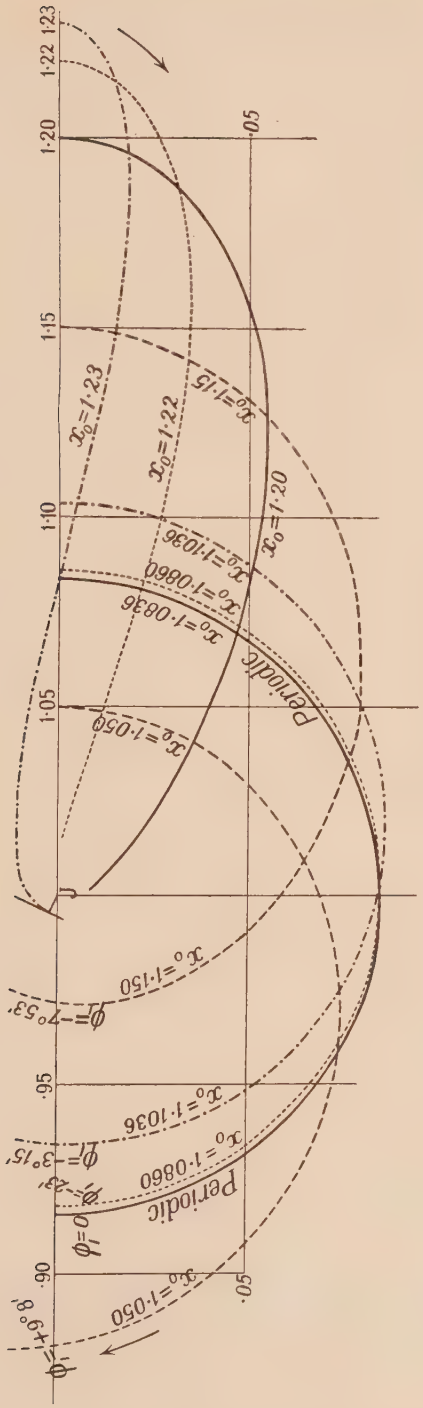


FIG. 1.—RETROGRADE SATELLITES, $C=40$.
STABLE PERIODIC ORBIT.

satellite which retains its retrograde character, with a very small pericentral distance. This portion of the orbit is not drawn, but

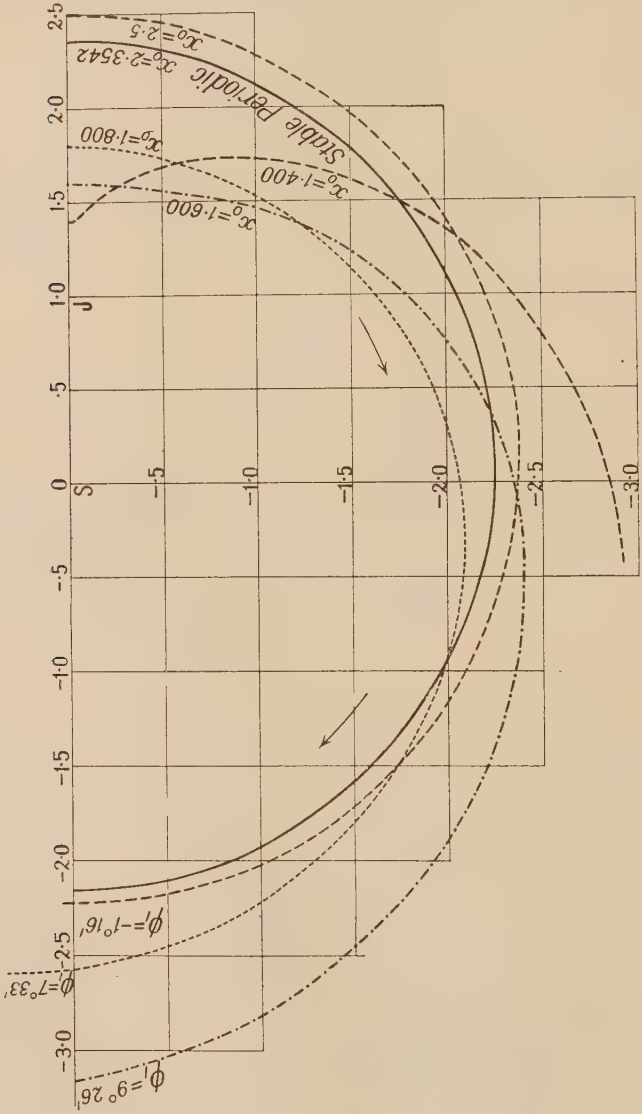


FIG. 2.—SUPERIOR PLANETS, DIRECT IN SPACE, RETROGRADE AS TO MOVING AXES; WHEN $C=3{\cdot}9$.

the pericentre is reached before the satellite reaches the line of syzygy, and ϕ_1, the angle of crossing that line, is negative.

As x_0 continues to diminish ϕ_1 increases, passes through zero, and when $x_0=1{\cdot}050$ rises to 9° 8′.

When $x_0 = 1{\cdot}08596$, $\phi_1 = -23'$, with a period given by $nT = 27^\circ\ 13'$; and when $x_0 = 1{\cdot}10364$, $\phi_1 = -3^\circ\ 15'$.

By extrapolation from these I find that $x_0 = 1{\cdot}0836$ will make $\phi_1 = 0$, and the orbit periodic, as shown in the figure by the firm line.

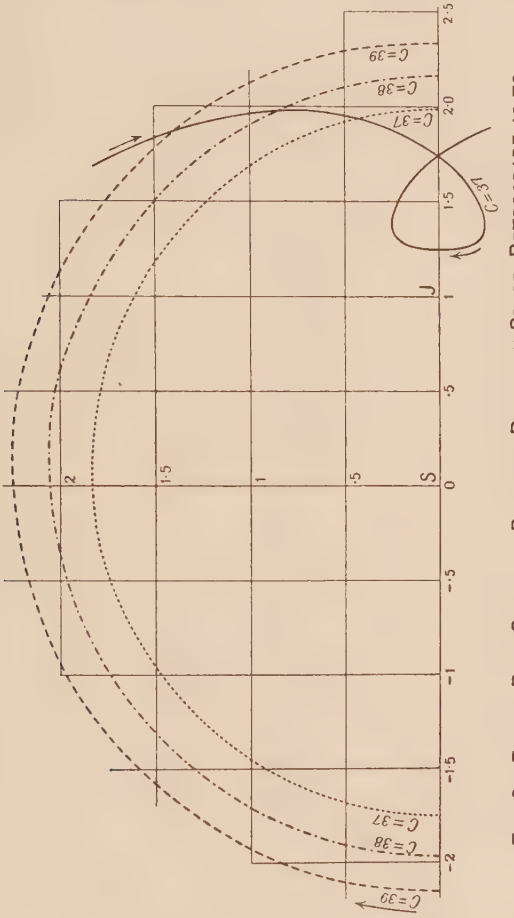


FIG. 3.—FAMILY D OF SUPERIOR PLANETS, DIRECT IN SPACE, RETROGRADE AS TO MOVING AXES; ALL STABLE. NON-PERIODIC $x_0 = 1{\cdot}25$, $C = 37$.

The orbit $x_0 = 1{\cdot}08596$ is so nearly periodic that it suffices for the computation of stability, and I find the orbit to be very stable. The numerical details will be given in an appendix to the reproduction in my collected papers.

§ 4. *Superior Planets.*

I began, as in my previous work, by computing a number of orbits corresponding to $C = 39$. They are exhibited in Fig. 2,

where J is at the point marked 1·0. Some of the computations for orbits of this kind have been lost, but results were saved which enabled me to draw the curve marked as periodic. The only curve to which it seems desirable to draw attention is that marked $x_0 = 1{\cdot}400$. In this curve we see initially a somewhat sharp curvature away from J. The curve is approximating to the shape which is assumed by an orbit of ejection from J away from S. It seems to indicate that the periodic orbits, of which one is shown in this figure, will end by becoming a heart-shaped orbit of ejection from J, and will then become retrograde. Another heart-shaped orbit will be referred to in § 6 below.

I next continued to investigate the periodic orbits of which one had been found, and determined the members of this family, which I call D, for $C = 39$, 38, 37, as shown in Fig. 3. These orbits are all stable, and details for their construction will be given in an appendix in the republication.

Fig. 3 also shows a remarkable form of looped orbit which I found with $C = 37$. It indicates, without much room for doubt, the existence of families of inverted figure-of-8 orbits, with the small loop inside the large one, as further explained in Part V. As in the case of the A satellites, and the A′, A″ figure-of-8 orbits (considered in Part IV.), there must, in connection with the D planets, be two such families, which may be called D′ and D″.

III. Orbits of Ejection and Cusps.

§ 5. *Formation of the Equations for Orbits of Ejection.*

Orbits in which the body is ejected either from Jove or from the Sun along the line of syzygies play an important part in the continuity of families of orbits; it is therefore convenient to find formulæ which will render it possible to trace such orbits for a considerable distance from their starting-point. We shall, of course, at the same time find how a body may fall into Jove or the Sun so as to move at the last tangentially to the line of syzygy.

The case to be considered in detail is where the ejection is from J towards S, and it will be easy to determine therefrom the equations for the three other cases.

The formulæ from which the solution will be found are those in § 1 of "Periodic Orbits," namely:—

$$\frac{1}{R} = \frac{P}{V^2} - \frac{2n}{V},$$

$$P = \nu\left(\frac{1}{r^2} - r\right)\cos(\phi - \theta) + \left(\frac{1}{\rho^2} - \rho\right)\cos(\phi - \psi),$$

$$V^2 = \nu\left(r^2 + \frac{2}{r}\right) + \left(\rho^2 - \frac{2}{\rho}\right) - C,$$

$$\phi = \phi_0 + \int \frac{ds}{R}, \quad x = x_0 - \int \sin\phi\, ds, \; y = y_0 + \int \cos\phi\, ds.$$

In the case to be considered $\phi_0 = \frac{1}{2}\pi$, $x_0 = 1$, $y_0 = 0$.

Since the body starts straight away from J, the arc s must be at first nearly equal to the radius vector ρ. This suggests development in powers of ρ.

The change of independent variable from s to ρ may be made by means of the formula $ds = -\operatorname{cosec}(\phi - \psi)d\rho$, which is obvious from geometrical considerations.

If at a finite distance from J the body were still moving exactly along SJ, P would be zero and $1/R$ would have the form $2n/V$. Since near J the radius vector r is nearly equal to unity, ρ is small, and V^2 is approximately equal to $2/\rho - (C - 3\nu)$. Hence, when $1/V$ is developed in powers of ρ, it must begin with $\rho^{\frac{1}{2}}$, and $1/R$ must be expressible by a series of the same form. Although P is not zero, this argument is found to have suggested the proper form for the series.

I assume, then,

$$\frac{1}{R} = -5a\rho^{\frac{1}{2}}[1 + A_1\rho + A_2\rho^2 + A_3\rho^3 + \dots].$$

The 5 is introduced for convenience, and at present I do not go beyond ρ^3.

It is first necessary to find ds in powers of ρ.

The perpendicular on to the tangent to the orbit is $\rho\cos(\phi - \psi)$ and a known formula gives

$$\frac{1}{R} = \frac{1}{\rho}\frac{d}{d\rho}[\rho\cos(\phi - \psi)].$$

Hence

$$\cos(\phi - \psi) = \int \frac{\rho}{R}d\rho$$

$$= -2a\rho^{\frac{3}{2}}[1 + \tfrac{5}{7}A_1\rho + \tfrac{5}{9}A_2\rho^2 + \tfrac{5}{11}A_3\rho^3 + \dots].$$

From this we find

$$\sin^2(\phi - \psi) = 1 - 4a^2\rho^3 + 4\tfrac{2\cdot5}{7}a^2A_1\rho^4 - 4a^2(\tfrac{2\cdot5}{9}A_2 + \tfrac{5^2}{7^2}A_1^2)\rho^5$$
$$- 4a^2(\tfrac{2\cdot5}{11}A_3 + 2.\tfrac{5}{7}.\tfrac{5}{9}A_1A_2)\rho^6 - \dots$$

Whence, taking the negative sign of the reciprocal of the square root,

$$\frac{ds}{d\rho} = -\operatorname{cosec}(\phi - \psi)$$
$$= 1 + 2a^2\rho^3 + 2\tfrac{2\cdot5}{7}a^2A_1\rho^4 + 2a^2(\tfrac{2\cdot5}{9}A_2 + \tfrac{5^2}{7^2}A_1^2)\rho^5$$
$$+ 2a^2(\tfrac{2\cdot5}{11}A_3 + 2.\tfrac{5}{7}.\tfrac{5}{9}A_1A_2 + 3a^2)\rho^6 + \dots$$

The determination of $1/R$ as far as ρ^3 gives $ds/d\rho$ as far as ρ^6, but at present it is only required as far as ρ^3.

Thus $ds = (1 + 2a^2\rho^3)d\rho$,

and $$\frac{ds}{R} = -5a\rho^{\frac{1}{2}}[1 + A_1\rho + A_2\rho^2 + (A_3 + 2a^2)\rho^3]d\rho .$$

By integration we obtain

$$\phi = \tfrac{1}{2}\pi - \tfrac{10}{3}a\rho^{\frac{3}{2}}[1 + \tfrac{3}{5}A_1\rho + \tfrac{3}{7}A_2\rho^2 + \tfrac{1}{3}(A_3 + 2a^2)\rho^3] .$$

The development of $\cos\phi$ and $\sin\phi$ may be effected in powers of $\frac{1}{2}\pi - \phi$; whence

$$\cos\phi = \tfrac{10}{3}a\rho^{\frac{3}{2}}[1 + \tfrac{3}{5}A_1\rho + \tfrac{3}{7}A_2\rho^2 + \tfrac{1}{3}(A_3 - 2\,.\,\tfrac{4^2}{3^2}a^2)\rho^3] ,$$

$$\sin\phi = 1 - 2\tfrac{5^2}{3^2}a^2\rho^3[1 + \tfrac{2.3}{5}A_1\rho + (\tfrac{2.3}{7}A_2 + \tfrac{3^2}{5^2}A_1^2)\rho^2 + (\tfrac{2}{3}A_3 + \tfrac{2.3^3}{5.7}A_1A_2 + \tfrac{11}{3^3}a^2)\rho^3] .$$

As yet, however, we only need $\sin\phi$ as far as ρ^3.
Thus

$$\cos\phi ds = \tfrac{10}{3}a\rho^{\frac{3}{2}}[1 + \tfrac{3}{5}A_1\rho + \tfrac{3}{7}A_2\rho^2 + \tfrac{1}{3}(A_3 + 2\,.\,\tfrac{11}{3^2}a^2)\rho^3]d\rho ,$$

$$\sin\phi ds = [1 - 2\,.\,\tfrac{4^2}{3^2}a^2\rho^3]d\rho .$$

Hence by integration

$$y = \tfrac{4}{3}a\rho^{\frac{3}{2}}[1 + \tfrac{3}{7}A_1\rho + \tfrac{5}{3.7}A_2\rho^2 + \tfrac{5}{3.11}(A_3 + 2\,.\,\tfrac{11}{3^2}a^2)\rho^3] ,$$

$$x = 1 - \rho + \tfrac{8}{3^2}a^2\rho^4 .$$

If we square these and add them together we obtain r^2. Since y^2 begins with ρ^5, r does not differ from x as far as the fourth power of ρ.
Accordingly $$r = 1 - \rho + \tfrac{8}{3^2}a^2\rho^4 .$$

We are now in a position to determine V^2, but I shall replace the constant C by another, E such that

$$C = 4E + 3\nu .$$

The formula for V^2 thus becomes

$$V^2 = \frac{2}{\rho}\left[1 + \tfrac{1}{2}\rho^3 + \tfrac{1}{2}\nu\rho\left(r^2 + \frac{2}{r}\right) - 2E\rho - \tfrac{3}{2}\nu\rho\right] .$$

On substituting for r its value in terms of ρ we obtain

$$V^2 = \frac{2}{\rho}[1 - 2E\rho + \tfrac{1}{2}(3\nu + 1)\rho^3 + \nu\rho^4 + \nu\rho^5] .$$

I have no present use for the terms in ρ^4 and ρ^5.
Since $n^2 = \nu + 1$, $\frac{1}{2}(3\nu + 1) = \frac{3}{2}n^2 - 1$. Hence, as far as necessary,

$$V^2 = \frac{2}{\rho}[1 - 2E\rho - (\tfrac{3}{2}n^2 - 1)\rho^3] .$$

Whence

$$\frac{1}{V^2} = \tfrac{1}{2}\rho\left[1 + 2E\rho + 4E^2\rho^2 + (8E^3 - \tfrac{3}{2}n^2 + 1)\rho^3\right],$$

$$\frac{1}{V} = \frac{1}{\sqrt{2}}\rho^{\frac{1}{2}}\left[1 + E\rho + \tfrac{3}{2}E^2\rho^2 + (\tfrac{5}{2}E^3 - \tfrac{3}{4}n^2 + \tfrac{1}{2})\rho^3\right].$$

In order to find $1/R$ it will be necessary to form $\nu\left(\frac{1}{r^2} - r\right)/V^2$ and $\left(\frac{1}{\rho^2} - \rho\right)/V^2$. As far as ρ^2,

$$\frac{1}{r^2} - r = 3\rho(1 + \rho),$$

and

$$\frac{\nu}{V^2}\left(\frac{1}{r^2} - r\right) = \tfrac{3}{2}\nu\rho^2\left[1 + (2E + 1)\rho\right], \text{ as far as } \rho^3. \quad . \quad (2)$$

This expression has to be multiplied by $\cos(\phi - \theta)$, and as it will only be required to develop $1/R$ as far as $\rho^{\frac{7}{2}}$, it is clear that it will only be necessary to use the development of $\cos(\phi - \theta)$ as far as $\rho^{\frac{3}{2}}$.

Now, $\sin\theta = y/r$, and the value of y found above shows that the first term in θ is $\frac{4}{3}\alpha\rho^{\frac{5}{2}}$. Hence to the order required $\theta = 0$, and

$$\cos(\phi - \theta) = \cos\phi = \tfrac{10}{3}\alpha\rho^{\frac{3}{2}}.$$

Writing $\nu = n^2 - 1$, we have to the required order,

$$\frac{\nu}{V^2}\left(\frac{1}{r^2} - r\right)\cos(\phi - \theta) = -5\alpha\rho^{\frac{1}{2}}\left[-(n^2 - 1)\rho^3\right]. \quad . \quad (3)$$

Since $\frac{1}{\rho^2} - \rho = \frac{1}{\rho^2}(1 - \rho^3)$,

$$\frac{1}{V^2}\left(\frac{1}{\rho^2} - \rho\right) = \frac{1}{2\rho}\left[1 + 2E\rho + 4E^2\rho^2 + (8E^3 - \tfrac{3}{2}n^2)\rho^3\right],$$

and

$$\cos(\phi - \psi) = -2\alpha\rho^{\frac{3}{2}}\left[1 + \tfrac{5}{7}A_1\rho + \tfrac{5}{9}A_2\rho^2 + \tfrac{5}{11}A_3\rho^3\right].$$

Whence

$$\begin{aligned}\frac{1}{V^2}\left(\frac{1}{\rho^2} - \rho\right)\cos(\phi - \psi) &= -5\alpha\rho^{\frac{1}{2}}\left[\tfrac{1}{5} + (\tfrac{2}{5}E + \tfrac{1}{7}A_1)\rho + (\tfrac{4}{5}E^2 + \tfrac{2}{7}EA_1 + \tfrac{1}{9}A_2)\rho^2\right. \\ &\quad \left. + (\tfrac{8}{5}E^3 - \tfrac{3}{10}n^2 + \tfrac{4}{7}E^2A_1 + \tfrac{2}{9}EA_2 + \tfrac{1}{11}A_3)\rho^3\right] \quad (4)\end{aligned}$$

Also

$$-\frac{2n}{V} = -n\sqrt{2}\,.\,\rho^{\frac{1}{2}}\left[1 + E\rho + \tfrac{3}{2}E^2\rho^2 + (\tfrac{5}{2}E^3 - \tfrac{3}{4}n^2 + \tfrac{1}{2})\rho^3\right]. \quad (5)$$

The sum of (3), (4), and (5) is equal to $1/R$.
If we consider only the term in $\rho^{\frac{1}{2}}$, we have

$$\frac{1}{R} = -\alpha\rho^{\frac{1}{2}} - n\rho^{\frac{1}{2}}\sqrt{2}\,.$$

But our assumed expression for $1/R$ was $-5\alpha\rho^{\frac{1}{2}}$. Hence $\alpha = \frac{1}{4}n\sqrt{2}$.

On substituting this value for α, the sum of (3), (4), and (5) gives us

$$\frac{1}{R} = -\tfrac{5}{4}n\sqrt{2}\,.\rho^{\frac{1}{2}}\left[1 + (\tfrac{6}{5}E + \tfrac{1}{7}A_1)\rho + (2E^2 + \tfrac{2}{7}EA_1 + \tfrac{1}{9}A_2)\rho^2 + (\tfrac{18}{5}E^3 - \tfrac{19}{10}n^2 + \tfrac{7}{5} + \tfrac{4}{7}E^2A_1 + \tfrac{2}{9}EA_2 + \tfrac{1}{11}A_3)\rho^3\right].$$

By equating the coefficients inside the bracket to A_1, A_2, A_3 respectively, we find

$$A_1 = \tfrac{7}{5}E\,,\quad A_2 = \tfrac{27}{10}E^2\,,\quad A_3 = \tfrac{11}{2}E^3 - \tfrac{2009}{1000}n^2 + \tfrac{77}{50}\,.$$

It is now easy to find the solution. Suppose that we write

$$\left.\begin{aligned} \phi &= \tfrac{1}{2}\pi - \tfrac{5}{6}n\sqrt{2}\,.\rho^{\frac{3}{2}}\left[1 + \Sigma B_i\rho^i\right],\\ y &= \tfrac{1}{3}n\sqrt{2}\,.\rho^{\frac{5}{2}}\left[1 + \Sigma C_i\rho^i\right],\\ x &= 1 - \rho + \Sigma D_i\rho^i,\\ \frac{ds}{d\rho} &= \left[1 + \Sigma\alpha_i\rho^i\right],\\ \frac{d\rho}{ds} &= \left[1 + \Sigma\beta_i\rho^i\right], \end{aligned}\right\} \quad . \quad . \quad (6)$$

then I find

$$B_1 = \tfrac{21}{25}E\,,\quad B_2 = \tfrac{81}{70}E^2\,,\quad B_3 = \tfrac{11}{6}E^3 - \tfrac{46}{75}n^2 + \tfrac{77}{150}\,;$$

$$C_1 = \tfrac{3}{5}E\,,\quad C_2 = \tfrac{9}{14}E^2\,,\quad C_3 = \tfrac{5}{6}E^3 - \tfrac{73}{270}n^2 + \tfrac{7}{30}\,;$$

$$D_2 = 0\,,\quad D_3 = 0\,,\quad D_4 = \tfrac{1}{9}n^2\,;$$

$$\alpha_1 = 0\,,\quad \alpha_2 = 0\,,\quad \alpha_3 = \tfrac{1}{4}n^2\,,\quad \alpha_4 = \tfrac{1}{2}En^2\,,\quad \alpha_5 = E^2n^2\,,\quad \alpha_6 = 2E^3n^2 - \tfrac{61}{160}n^4 + \tfrac{7}{20}n^2\,;$$

$$\beta_1 = 0\,,\quad \beta_2 = 0\,,\quad \beta_3 = -\tfrac{1}{4}n^2\,,\quad \beta_4 = -\tfrac{1}{2}En^2\,,\quad \beta_5 = -E^2n^2\,,\quad \beta_6 = -(2E^3n^2 - \tfrac{157}{320}n^4 + \tfrac{7}{20}n^2)\,.$$

In order to find the time we may proceed from the formula

$$nt = n\int \frac{1}{V}\frac{ds}{d\rho}d\rho\,.$$

This involves the multiplication together of two series and subsequent integration. The series for $1/V$ and $ds/d\rho$ are given

above, and if we write

$$nt = \tfrac{1}{3}\sqrt{2}\,.\,n\rho^{\frac{3}{2}}\left[1 + \Sigma G_i \rho^i\right],$$

we find

$$G_1 = \tfrac{3}{8}E,\ G_2 = \tfrac{9}{14}E^2,\ G_3 = \tfrac{5}{6}E^3 - \tfrac{1}{6}\nu\,.$$

Since we now have $ds/d\rho$ as far as ρ^6, we have the means of determining A_4, A_5, A_6. Thus we multiply $1/R$ by ds, and find ϕ as far as $\rho^{\frac{3}{2}}\rho^6$; $\cos(\phi - \psi)$ may be found by integrating ρ/R. In the further development of $\cos(\phi - \theta)$, it is useful to use the formula

$$r\cos(\phi - \theta) = \cos\phi + \rho\cos(\phi - \psi)\,.$$

Finally, by following the same procedure as before we find equations from which A_4, A_5, A_6 may be found, and thence the successive B's, C's, D's. The analysis is tedious, and the results will merely be given in a tabular form. I shall moreover omit A_6, B_6, C_6, since the expressions become very long.

Thus far we have considered ejection from J towards S, and we will now consider ejection from J away from S.

The same forms of series serve again, but the constant of integration in finding ϕ becomes $-\frac{1}{2}\pi$. Hence $\sin\phi$ and $\cos\phi$ simply have their signs changed. When these are multiplied by ds and integrated, y is found to have the same form as before, but with opposite sign. However, x becomes $1 + \rho - \frac{8}{9}a^2\rho^4$. Hence it follows that

$$V^2 = \frac{2}{\rho}\left[1 - 2E\rho + (\tfrac{3}{2}n^2 - 1)\rho^3 - (n^2 - 1)\rho^4 + (n^2 - 1)\rho^5\right].$$

In determining A_1, A_2, A_3 the last two terms are not needed, so that V^2 has the same form as before.

The expression $\frac{1}{r^2} - r$ becomes -3ρ, so that its sign is changed. But $\cos(\phi - \theta)$ also has a change of sign, so that $\frac{1}{V^2}\left(\frac{1}{r^2} - r\right)\cos(\phi - \theta)$ remains as before.

The form of $\cos(\phi - \psi)$ clearly remains the same.

Thus as far as ρ^3 the result is obtained by simply changing certain signs. Further on, however, other changes are introduced.

Ejection from S may be treated by symmetry. Ejection towards J is symmetrical with ejection towards S. It is clear that r replaces ρ and ρ replaces r; $-y$ replaces y; $1 - x$ replaces x; π must be subtracted from ϕ; the Sun replaces Jove, and *vice versâ*, so that $1/\nu$ replaces ν, and hence n^2/ν replaces n^2. The n which occurs in the several coefficients at the beginning of each series represents the angular velocity of the axes and remains unchanged. The only difficulty is in the constant E or $\frac{1}{4}(C - 3\nu)$, and it will be found that this must be replaced by $F = \frac{1}{4}(C - 3)/\nu$.

The following table gives the results:—

ORBITS OF EJECTION FROM J.

The upper signs refer to ejection towards S, the lower away from S.

$$\frac{1}{R} = -\tfrac{5}{4}n(2\rho)^{\frac{1}{3}}\left[1 + \Sigma A_i\rho^i\right],$$

$$E = \tfrac{1}{4}(C - 3\nu).$$

$$A_1 = \tfrac{7}{5}E; \quad A_2 = \tfrac{27}{10}E^2; \quad A_3 = \tfrac{11}{2}E^3 - \tfrac{209}{100}\nu - \tfrac{11}{20};$$
$$A_4 = \tfrac{91}{8}E^4 - \tfrac{21119}{3000}E\nu - \tfrac{39}{20}E \mp \tfrac{13}{6}\nu;$$
$$A_5 = \tfrac{189}{8}E^5 - \tfrac{3871}{1960}E^2\nu - \tfrac{45}{8}E^2 \mp \tfrac{481}{70}E\nu - \tfrac{41}{14}\nu.$$

N.B.—The coefficient of the highest power of E is $\frac{1}{6}(2i+5)\left(\frac{2i!}{2^i(i!)^2}\right)$.

$$\phi = \pm\tfrac{1}{2}\pi - \tfrac{5}{6}n\sqrt{2}.\rho^{\frac{3}{2}}\left[1 + \Sigma B_i\rho^i\right],$$

$$B_1 = \tfrac{21}{25}E; \quad B_2 = \tfrac{81}{70}E^2; \quad B_3 = \tfrac{11}{6}E^3 - \tfrac{466}{75}\nu - \tfrac{1}{10};$$
$$B_4 = \tfrac{2738}{88}E^4 - \tfrac{4666}{2475}E\nu - \tfrac{3}{10}E \mp \tfrac{13}{252}\nu;$$
$$B_5 = \tfrac{5674}{104}E^5 - \tfrac{1277121}{31850}E^2\nu - \tfrac{3}{4}E^2 \mp \tfrac{111}{70}E\nu - \tfrac{123}{882}\nu.$$

$$y = \pm\tfrac{1}{3}n\sqrt{2}.\rho^{\frac{3}{2}}\left[1 + \Sigma C_i\rho^i\right],$$

$$C_1 = \tfrac{3}{5}E; \quad C_2 = \tfrac{9}{14}E^2; \quad C_3 = \tfrac{5}{6}E^3 - \tfrac{73}{270}\nu - \tfrac{1}{27};$$
$$C_4 = \tfrac{105}{88}E^4 - \tfrac{199}{3300}E\nu - \tfrac{1}{15}E \mp \tfrac{5}{252}\nu;$$
$$C_5 = \tfrac{189}{104}E^5 - \tfrac{76313}{637000}E^2\nu - \tfrac{39}{350}E^2 \mp \tfrac{37}{70}E\nu - \tfrac{41}{1820}\nu.$$

$$x = 1 \mp \rho \pm \Sigma D_i\rho^i,$$

$$D_2 = 0; \quad D_3 = 0; \quad D_4 = \tfrac{1}{9}n^2; \quad D_5 = \tfrac{2}{15}En^2; \quad D_6 = \tfrac{32}{175}E^2n^2;$$
$$D_7 = n^2\left[\tfrac{2564}{945}E^3 - \tfrac{1316}{24300}\nu - \tfrac{1}{4860}\right];$$
$$D_8 = n^2\left[\tfrac{20488}{48512}E^4 - \tfrac{34588}{222775}E\nu - \tfrac{2}{405}E - \tfrac{5}{99}\nu\right];$$
$$D_9 = n^2\left[\tfrac{20488}{300032}E^5 - \tfrac{16883998}{4729725}E^2\nu - \tfrac{466}{47225}E^2 \mp \tfrac{5126}{34650}E\nu - \tfrac{41}{8190}\nu\right].$$

$$ds = d\rho\left[1 + \Sigma\alpha_i\rho^i\right],$$

$$\alpha_1 = 0; \quad \alpha_2 = 0; \quad \alpha_3 = \tfrac{1}{4}n^2; \quad \alpha_4 = \tfrac{1}{2}En^2; \quad \alpha_5 = E^2n^2;$$
$$\alpha_6 = n^2\left[2E^3 - \tfrac{61}{1600}\nu - \tfrac{1}{32}\right].$$

$$d\rho = ds\left[1 + \Sigma\beta_i\rho^i\right],$$

$$\beta_1 = 0; \quad \beta_2 = 0; \quad \beta_3 = -\tfrac{1}{4}n^2; \quad \beta_4 = -\tfrac{1}{2}En^2;$$
$$\beta_5 = -E^2n^2; \quad \beta_6 = -n^2\left[2E^3 - \tfrac{1570}{3240}\nu - \tfrac{9}{64}\right].$$

$$nt = \tfrac{1}{3}n\sqrt{2}.\rho^{\frac{3}{2}}\left[1 + \Sigma G_i\rho^i\right],$$

$$G_1 = \tfrac{3}{5}E; \quad G_2 = \tfrac{9}{14}E^2; \quad G_3 = \tfrac{5}{6}E^3 - \tfrac{1}{6}\nu; \quad G_4 = \tfrac{105}{88}E^4 - \tfrac{3}{22}\nu(3E \pm 1);$$
$$G_5 = \tfrac{189}{104}E^5 - \nu\left(\tfrac{45}{32}E^2 \pm \tfrac{9}{26}E + \tfrac{3}{26}\right).$$

The same formulæ may be used for ejection from S, with the appropriate changes indicated above.

§ 6. *On certain Orbits of Ejection.*

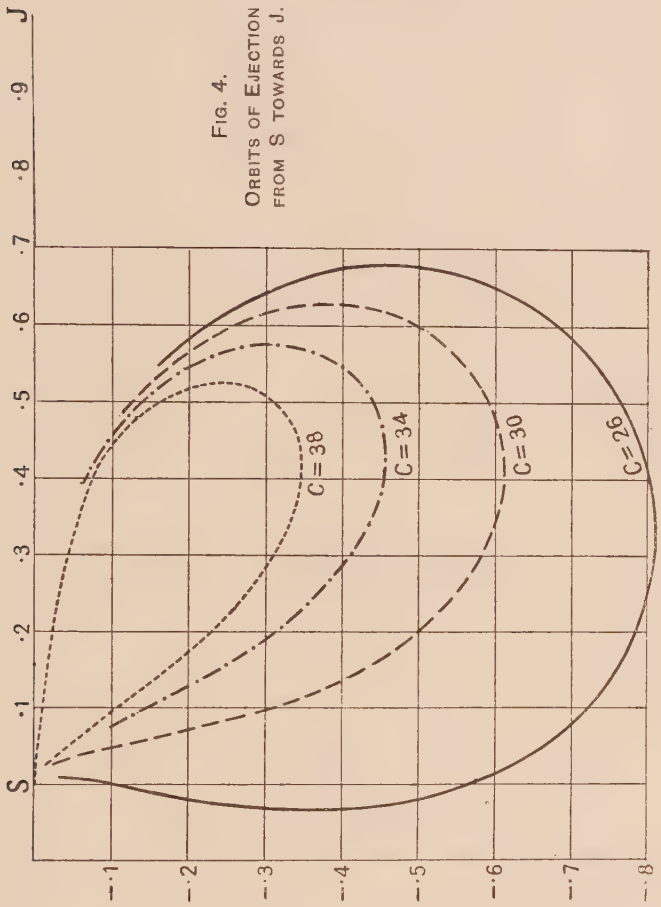


FIG. 4.
ORBITS OF EJECTION FROM S TOWARDS J.

This class of orbits was considered principally with the object of completing the classification of the family A of satellites, as indicated in the Introduction. It will therefore be more convenient to consider the orbits of ejection from J towards S in Part IV. below. We will here, therefore, only consider a group of orbits arising from ejection from S towards J. Several of these orbits are shown in Fig. 4, and we may see the gradual transformation which occurs as the constant of relative energy falls.

Being curious to understand the later development for smaller values of C, I computed at hazard the ejectional orbit for $C = 20$, and happened by chance on a very curious form of periodic orbit.

This is exhibited in Fig. 5, and it will be observed that the body is not only ejected along the line of syzygy, but returns to S tangentially to the same line with extraordinary exactness. There is of course another branch, on the positive side of the line of syzygy but not shown in the figure, in which the ejection takes place away from J and where the return is on the side towards J. It seemed worth while to show how the body moves in space whilst performing this curious orbit with respect to moving axes, and the result is exhibited in an inset of the same figure. The body starts in a straight line towards J, and is gradually slightly deflected by J. By the time it has reached and passed the place where J was, J has passed away, and it falls back on S nearly in a straight line, so timing its fall that the bodies are again in a

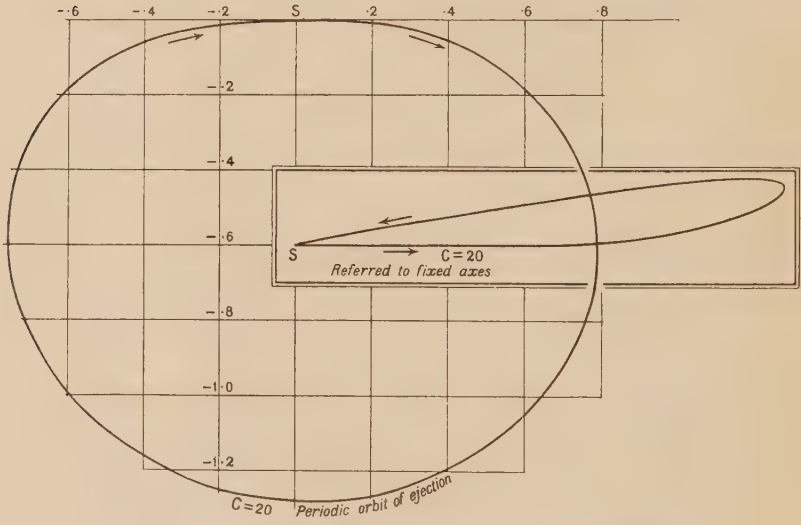


FIG. 5. A PERIODIC ORBIT OF EJECTION FROM S TOWARDS J.

straight line as it reaches the Sun again. The time from ejection to return is obviously given by a value of nT a little greater than 180°.

It is easy to perceive that this orbit is the limiting form of four others which coalesce. A body may move in any one of the four following ways, all coalescent in the orbit in Fig. 5 :—

(1) Initially direct, finally direct.
(2) Initially direct, finally retrograde.
(3) Initially retrograde, finally direct.
(4) Initially retrograde, finally retrograde.

It is probable that the possibility of the simultaneous coalescence of these four kinds of orbits depends on the choice of a particular

value of the Sun's mass, in this case ten times that of the large planet Jove. I shall give the numerical values for Fig. 5 in my republication.

In the course of the further investigation of this class of orbit I computed one more, which proves the existence of a heart-shaped periodic orbit of ejection from S towards J.

When $C=3$ the orbit was found to cut the line of syzygies where $x=-3{\cdot}57$, $y=0$, and $\phi=11\frac{1}{2}^\circ$. Now when C was 20 the final ϕ was -90°, and therefore for a value of C a very little greater than 3 the final ϕ must vanish, and the orbit must be periodic. A figure exhibiting the results of the calculation shows that in this periodic orbit the crossing of the line of syzygy will occur when x is about equal to -3. A little consideration will show the reader that the orbit is heart-shaped, and that the body will fall back on S tangentially to the line of syzygy in the direction opposite from that along which it was ejected. It would be possible to give a fairly accurate drawing of this orbit, but it hardly seems worth while to do so, because Dr. Carl Burrau has already given illustrations of this kind of orbit, and has shown that they are probably limiting forms of the families of oscillating satellites.*

§ 7. *On the Form of a Cusp in an Orbit.*

If the third body reaches the curve of zero velocity the orbit has a cusp, and the common tangent to the two branches of the orbit is at right angles to the curve of zero velocity.

If the point x_0, y_0 (or in polar co-ordinates r_0, θ_0 or ρ_0, ψ_0) lies on the curve $2\Omega=C$, and if ϕ_0 defines the direction of the normal to that curve, we have

$$\nu\left(\frac{1}{r_0^2}-r_0\right)\cos(\phi_0-\theta_0)+\left(\frac{1}{\rho_0^2}-\rho_0\right)\cos(\phi_0-\psi_0)=0.$$

I find that the form of the orbit at this point is given by equations of which I shall not give the proof.

In order to express the result the following notation is adopted:—

$$f=\nu\left(\frac{1}{r_0^2}-r_0\right)\sin(\phi_0-\theta_0)+\left(\frac{1}{\rho_0^2}-\rho_0\right)\sin(\phi_0-\psi_0),$$

$$g=\frac{\nu}{r_0^3}+\frac{1}{\rho_0^3},$$

$$h=\frac{\nu}{r_0^3}\sin^2(\phi_0-\theta_0)+\frac{1}{\rho_0^3}\sin^2(\phi_0-\psi_0),$$

$$k=\frac{\nu}{r_0^3}\sin(\phi_0-\theta_0)\cos(\phi_0-\theta_0)+\frac{1}{\rho_0^3}\sin(\phi_0-\psi_0)\cos(\phi_0-\psi_0);$$

also s denotes the arc of orbit measured from the cusp, and ϕ_0 is

* *Ast. Nach.*, No. 3251, vol. 136 (1894).

the inclination of the tangent of the curve $2\Omega = 0$ to the x axis, and therefore ϕ_0 gives the initial value of ϕ in the orbit which is at right angles to the curve of zero velocity.

Then I find

$$\frac{1}{R} = \frac{k}{f} - \tfrac{1}{2}n\left(\frac{2}{f}\right)^{\frac{1}{3}}\left(1 - \frac{15h-7g}{4f}s\ldots\right),$$

$$\phi = \phi_0 - n\left(\frac{2s}{f}\right)^{\frac{1}{2}} + \frac{k}{f}s + \tfrac{1}{3}n\left(\frac{2}{f}\right)^{\frac{1}{3}}\left(\frac{15h-7g}{4f}\right)s^{\frac{3}{2}}\ldots,$$

$$x = r_0\cos\theta_0 + \cos\phi_0\left[\tfrac{2}{3}n\left(\frac{2}{f}\right)^{\frac{1}{3}}s^{\frac{3}{2}} - \frac{k}{2f}s^2 - \tfrac{2}{15}n\left(\frac{2}{f}\right)^{\frac{1}{3}}\frac{15h-7g+4n^2}{4f}s^{\frac{5}{2}}\ldots\right]$$
$$- \sin\phi_0\left[s - \frac{n^2}{2f}s^2 + \tfrac{2}{5}\frac{nk}{f}\left(\frac{2}{f}\right)^{\frac{1}{3}}s^{\frac{3}{2}}\ldots\right],$$

$$y = r_0\sin\theta_0 + \sin\phi_0\left[\tfrac{2}{3}n\left(\frac{2}{f}\right)^{\frac{1}{3}}s^{\frac{3}{2}} - \frac{k}{2f}s^2 - \tfrac{2}{15}n\left(\frac{2}{f}\right)^{\frac{1}{3}}\frac{15h-7g+4n^2}{4f}s^{\frac{5}{2}}\ldots\right]$$
$$+ \cos\phi_0\left[s - \frac{n^2}{2f}s^2 + \tfrac{2}{5}\frac{nk}{f}\left(\frac{2}{f}\right)^{\frac{1}{3}}s^{\frac{3}{2}}\ldots\right].$$

For small values of s the curve is a semicubical parabola.

These formulæ were of some use in the investigations of Part IV.

IV. The Families of Orbits of Satellites denoted A, A′, A″.

§ 8. *Completion of the Classification.*

A comparison of the orbit of the family C of satellites when $C = 38{\cdot}75$, as shown in Fig. 3, Plate IV. of my paper on "Periodic Orbits," with the orbit of the family A when $C = 40$, as shown in fig. 1, Plate IV., ought to have afforded a sufficient indication of the fate of the A satellites; it ought also to have indicated the existence of the A″ family of figure-of-8 orbits. Now that I know the result, the indication is so clear that I find it hard to understand how it came to be overlooked.

The A satellites are clearly tending to the ejectional form, and this was pointed out by Mr. Hough. Accordingly, in order to investigate the fate of the A family it was necessary to find an orbit of ejection which should cut the line of syzygy at right angles. This would form the end of that part of the family A in which the revolution round J is direct.

Secondly, it was required to find an orbit of ejection which should cut the line of syzygy at an angle a little less than π, and from this intersection the satellite should pass on, describe half a loop round the point of zero force, and end by cutting the line of syzygy a second time at right angles. This would form the end of the A″ orbits considered as direct orbits.

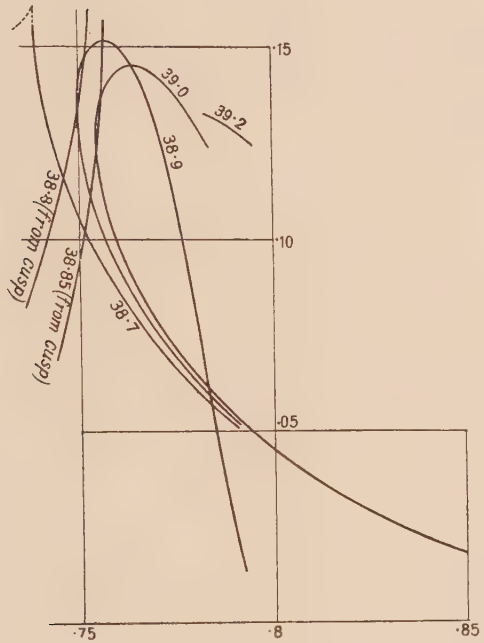


FIG. 6.—SOME ORBITS OF EJECTION FROM J TOWARDS S.

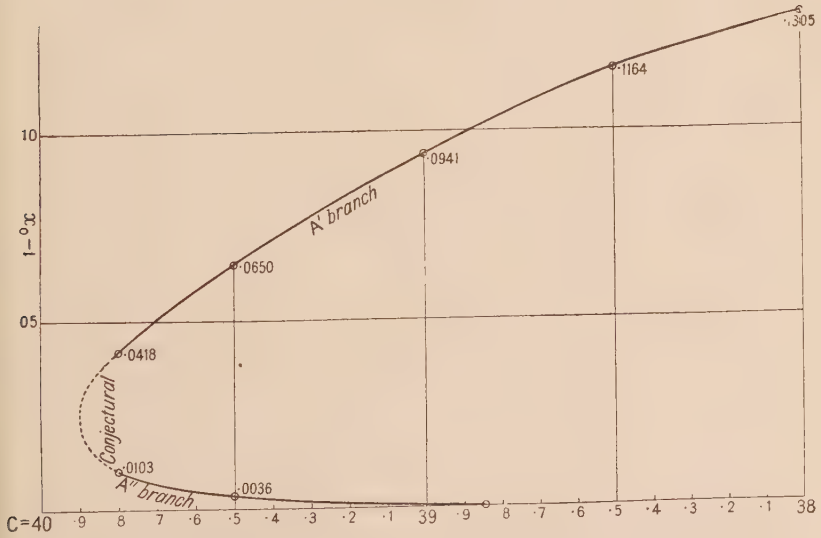


FIG. 7.—CURVE SHOWING COALESCENCE OF THE A′ AND A″ FAMILIES OF FIGURE-OF-8 ORBITS. COALESCENCE OCCURS FOR $x_0 = 1{\cdot}025$, $C = 39{\cdot}9$.

With these objects in view a great many orbits of ejection were computed. Unfortunately, I had become convinced that the critical orbit would be found to occur for a value of C very nearly equal to 40, and as this idea proved to be incorrect a great deal of my work became almost useless. However, in Fig. 6 I exhibit several of these orbits from which the results may be inferred with close accuracy.

A comparison of the orbit when $C = 39$ with the small portion of the orbit for $C = 39{\cdot}2$ shows that C had to be diminished further. The orbit for $C = 38{\cdot}9$ shows that C is still a very little too large, but that for $C = 38{\cdot}7$ it is too small. I conjecture that when C is about 38·85 the downward branch of the curve would cut the line of syzygy at right angles, and this would give us the ejectional form of the A family of satellites.

Again, if we were to take C a very little less than 38·85, say perhaps 38·845, the curve would cut the line of syzygies at an angle a very little less than π. Now, all the upper part of the loop would hardly be sensibly changed by these small changes in C. Hence we know with pretty close accuracy all the first part of the A″ orbit of ejection. In order to construct the loop of this orbit, it is best to begin from the left-hand side. The table of values for the oscillating satellite (a) when $C = 39$ are given on pp. 235–6 of my previous paper. We know that for $C = 38{\cdot}845$ the loop must fall just outside the path of this oscillating satellite. Hence by drawing a curve just outside this orbit (a), and joining it with nearly a straight line to the upper part of the loop of $C = 38{\cdot}9$, we shall obtain a very close representation of the ejectional form of the A″ orbit. The curve for $C = 38{\cdot}845$ in Fig. 9 below was constructed in this way, but it is marked as corresponding to 38·85 because, after all, we only know the value to be somewhat less than 38·9.

This result gives us the end of the A″ series, considered as directly moving satellites. It is yet more interesting to determine the beginning.

The A″ family of figure-of-8 orbits must arise from a form coalescent with the A′ family of figure-of-8 orbits, erroneously classified in my previous paper as belonging to the A family of satellites. In that paper I had the A′ family for $C = 38$, 38·5, 39, 39·5. It is therefore possible to construct a figure in which the abscissæ are C and the ordinates $x_0 - 1$. The upper curve in Fig. 7, as far as $C = 39{\cdot}5$ and $x_0 - 1 = {\cdot}0650$, shows the result as far as explained as yet; and one single point, viz. $C = 38{\cdot}85$ and $x_0 - 1 = 0$, has been obtained in the lower curve in Fig. 7.

It then became necessary to find another point on the lower curve, namely that for $C = 39{\cdot}5$. After some very troublesome calculations I found that if we start the third body at $x_0 - 1 = {\cdot}0036$, so that it is no longer quite an orbit of ejection, we shall obtain a figure-of-8 orbit which belongs to the family A″.

These two branches for A' and A'' have to be joined by a curve of rather sharp flexure. I made the junction conjecturally, and guessed that it would occur very nearly when $C = 39{\cdot}8$. It

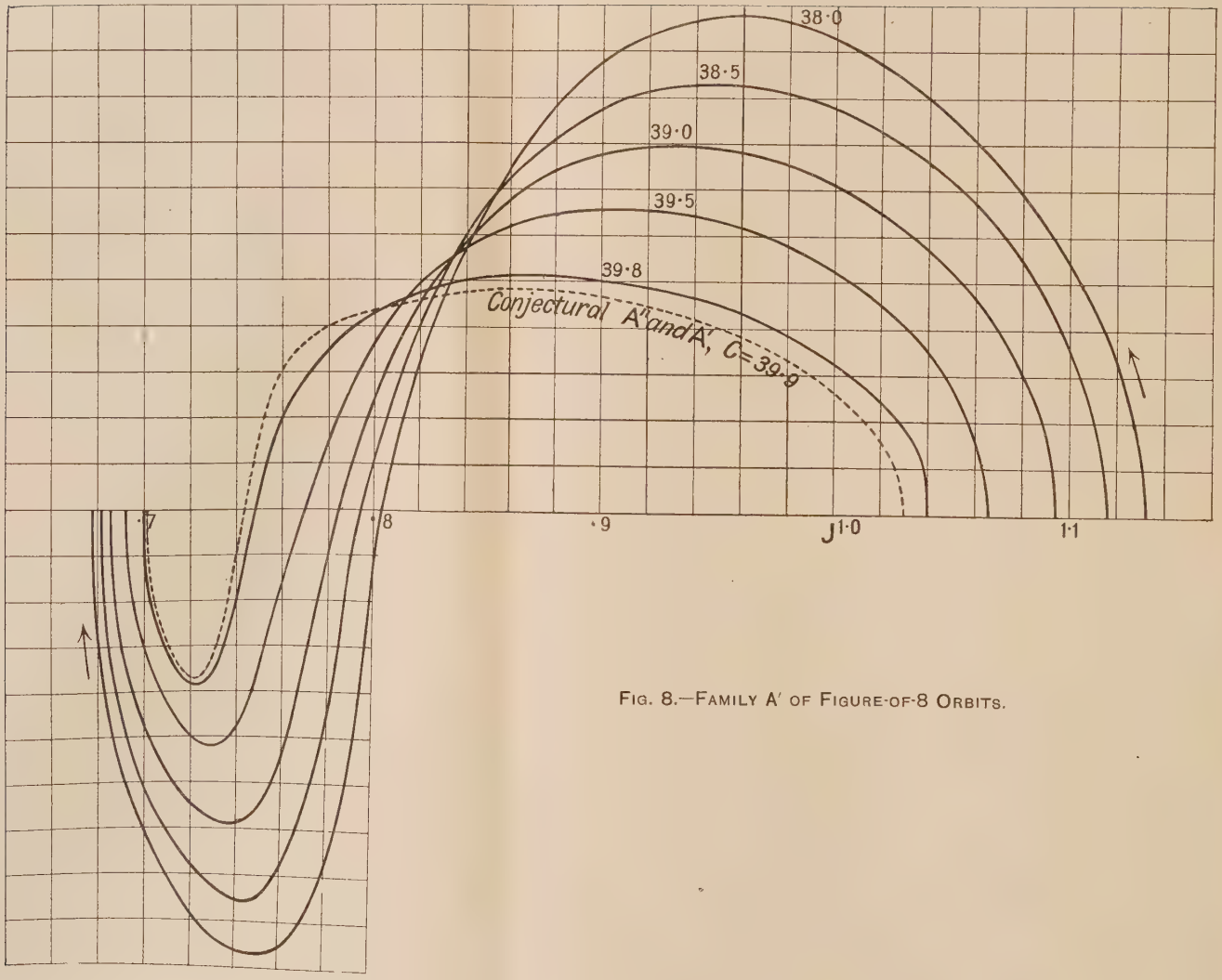


FIG. 8.—FAMILY A′ OF FIGURE-OF-8 ORBITS.

remained then to draw the two figures-of-8 orbits for this value of C. A great many tedious computations were made, whereby I found that the A′ orbit for this value of C will begin with $x_0 = 1{\cdot}0418$, and the A″ with $x_0 = 1{\cdot}0103$. These values give one point more on each of the two branches, and it appeared that the guess of 39·8 as the critical value was not a very good one. How-

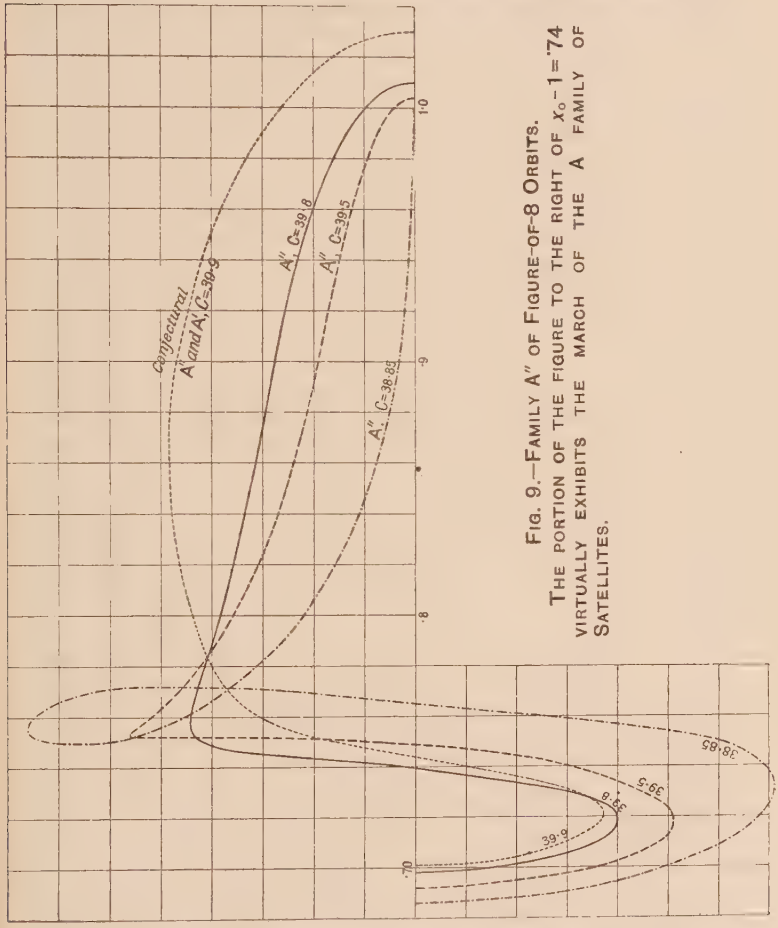


FIG. 9.—FAMILY A″ OF FIGURE-OF-8 ORBITS.
THE PORTION OF THE FIGURE TO THE RIGHT OF $x_0 - 1 = {\cdot}74$ VIRTUALLY EXHIBITS THE MARCH OF THE A FAMILY OF SATELLITES.

ever, it did not seem worth while to proceed to a new conjecture, and the two branches are joined by a dotted curve from which we may conclude with fair accuracy that the critical coalescent figure-of-8 orbit would be found for $C = 39{\cdot}9$ and $x_0 - 1 = {\cdot}025$.

Fig. 8, in which the four outer curves are taken from my previous paper, shows the series of the A′ family. It begins with the conjectural coalescent form when C is 39·9, and continues until C is 38; the starting-point continually recedes from J.

In Fig. 9 we have the A″ family beginning with the same coalescent form for $C = 39{\cdot}9$, and continuing until C has fallen to $38{\cdot}85$, when the orbit has assumed the ejectional form; after this the orbit becomes retrograde. During these changes the starting-point continually approaches J.

In the families A′ and A″ the third body describes circuits alternately about J and about the point of zero force. If it moves exactly on one of those curves it may perform these alternate circuits for ever, but otherwise it will end by moving off towards the neighbourhood of S or towards J. Now, for all starting-points on the line of syzygy between those of A′ and of A″ the body will ultimately or shortly move off towards the neighbourhood of S. Hence these prescribe the limiting forms of the paths of bodies which escape towards the Sun.

It is obvious that the third body might describe one circuit round J and then two round the point of zero force, and thus there must be another pair of limiting orbits, say A‴ and A^{IV}, in which the small loop of the 8 is described twice. The range of starting-points between A‴ and A^{IV} will be wider than in the case of A′ and A″. So further there must be other pairs of orbits in which the small loop is described, three, four, and more times, and there will be an increasing range between the starting-points.

The neck of the hour-glass figure of the curves of zero velocity first opens when $C = 40{\cdot}1821$, but the A′ and A″ orbits only first appear when C has fallen to $39{\cdot}9$. When the neck of the hour-glass has only a very small opening there will be a pair of figure-of-8 orbits which we may call $A^{(2n+1)}$, $A^{(2n+2)}$, in which the body moves once round J and then describes n circuits in the very small loop of the 8, and n will be large. These will be limiting forms of orbits in which the body can pass through the narrow neck of the hour-glass and escape to the Sun. As C falls in value the escape becomes easier and n falls, until finally when $C = 39{\cdot}9$ it has been reduced to zero, giving us the A′ and A″ families coalescent for that value. It must now be obvious how each successive pair of multiple figures-of-8 springs into existence as C falls, until at last the simplest pair, which we have traced, make their appearance. As Mr. Hough has pointed out, these orbits afford an interesting illustration of M. Poincaré's asymptotic orbits.

The sequence of changes in the A family of orbits of satellites is also virtually exhibited in Fig. 9, for considering only the portion of this figure to the right of $x = 1{\cdot}74$, we may imagine the descending portions of the curves to be shifted slightly to the right until they cut the line of syzygy at right angles; and this change will not affect the rest of the right-hand portion of the figure to a sensible extent. Thus Fig. 9 shows us that the members of the A family assume the form of a bell, first with rounded corners, then with sharp corners, then with loops. Finally these orbits pass through the ejectional stage and become retrograde.

A similar series of changes is to be observed in the C family of satellites, as shown in Fig. 3, Plate IV. of my previous paper. The

only difference is that whereas for C the mouth of the bell is directed away both from J and from S, in the A family it is directed away from J and towards S.

In my collected papers I shall give the numerical values from which these figures were drawn.

Of the two orbits A′ and A″ near coalescence, one must, according to Poincaré's principle, be stable and the other unstable, and it remains to distinguish between them. I have reason to believe that it is A′ which is unstable, and therefore A″ must be stable.

When the curvature of an orbit is abrupt both methods of discussing stability become very intractable. As the curvature of the A′ curve is rather less sharp than that of A″ it was chosen for treatment.

It was shown on p. 208 of "Periodic Orbits" that when $C = 39$ the function $\Delta \sin {}^{2}\frac{1}{2}\pi\sqrt{\Phi_0}$ for the A′ orbit was negative, and that the orbit was unstable, according to the class called even instability. In that case the representation of Φ by the Fourier series was not very satisfactory, yet sufficiently so, I think, to establish the conclusion. When the determinantal method was applied to the orbit A′ for $C = 39{\cdot}8$, the representation of Φ by the Fourier series was very wild, and the result can hardly be relied on further than as giving the sign of the determinant with some probability of correctness. I found it to be negative, and the instability was therefore probably of the same class as that in the case of $C = 39$.

When Mr. Hough's method was applied I found the same negative sign, indicating that c was of the form $\alpha\sqrt{-1}$, but the numerical value differed much from that found from the determinant.

It seems, then, almost certain that A′ is unstable near coalescence, and if this is so A″ must be stable. The margin of stability is, however, probably very small, and indeed it is possible that by the time C has fallen from $39{\cdot}9$ to $39{\cdot}8$ both orbits have become unstable.

V. Review of the Present Investigation.

§ 9. *Indications as to the Development of Old and the Existence of New Families.*

Several conclusions may be drawn from the results obtained above, of which some are certain and others rather speculative.

The families B and C of satellites were traced from their coalescent origin when C was $39{\cdot}3$ until they came to differ very widely from one another. In Part IV. the close analogy was pointed out between the development of the A family, as now traced, with that which was found before for the C family. The origin of these two families is, however, very different, for the A family begins with an infinitely small circle round J, and as the orbit expands with falling value of the constant C, it gradually loses its circularity; whereas B springs into existence along with the family C when the constant has fallen to $39{\cdot}3$.

In Part IV. the relationship has been traced between the A

satellites and the A′ and A″ figure-of-8 orbits, and we are now able to see that the existence of C will connote two families of figures-of-8, C′ and C″, with the smaller loop of the 8 directed away from S. Indeed, a fairly accurate drawing of one of these might be constructed from Fig. 1, Plate III. of "Periodic Orbits," where the orbit C is shown alongside of the oscillating satellite (*b*); for it is only necessary to make a connection crosswise between these two. I believe that this orbit would be of the family C″.

It is not as yet at all clear what is the nature of the future development of the family B, when the constant *C* falls below 38. From the succession of figures in "Periodic Orbits" it is obvious that the orbit B comes to agree more and more closely with the larger loop of the orbit A′, and that the right-hand side of the orbit of the oscillating satellite (*a*) is separated by an ever-diminishing space from the orbit B. May we not presume that B, A′, and (*a*) stretch out until they all draw near to the apex of the equilateral triangle drawn on SJ as a base? But I do not hazard any conjecture as to the changes beyond that stage.

We may feel confident that the family A of planets * must pass through changes exactly analogous to those of A, but the constant *C* will have to fall considerably below 38 before conspicuous changes set in. Thus the A planets must become retrograde by passing through the ejectional stage with ejection from S towards J. It would seem from an inspection of Fig. 4 that this stage is nearly reached when $C=26$. There will probably also be a pair of families of figure-of-8 orbits bearing the same relationship to the planetary A that A′ and A″ bear to the satellite A.

Presumably a pair of planetary orbits B and C will spring from coalescence, of which C will at first be stable. The planetary orbit C will lose its stability and will cut SJ nearer and nearer to S, while stretching outwards toward the point of zero force remote from J. This orbit must then become bell-shaped, ejectional, and then retrograde. In correlation with the planetary family C two new planetary figures-of-8 orbits must arise with the smaller loops circling the point of zero force on the side of S remote from J, and they must pass through changes analogous to those of the satellites C′ and C″. The point of zero force remote from J is so far distant from both bodies that all these changes will be much less conspicuous than in the cases already considered.

In this paper three members are traced in Fig. 3 of the family D of superior planets, which move directly in space but are retrograde with respect to the rotating axes. As the value of *C* falls the size of these orbits diminishes. I do not venture to make any suggestions as to their future development.

In Fig. 3 there is also shown a portion of a non-periodic orbit. The loop described in this case gives a fairly certain indication of the existence of two families of orbits, say *D′* and *D″*, which are analogous to figure-of-8 orbits, but have the smaller loop of the 8

* I shall be able to add the member of this family corresponding to $C=37{\cdot}5$ in republishing 'Periodic Orbits' in my collected papers.

inside the larger one. We may guess that one of these two becomes retrograde by passing through the stage of ejection from J towards S. Then again there must be two others of these inverted figures-of-8 connected with the point of zero force remote from J.

Possibly all these figures stretch themselves out towards the apices of the equilateral triangles erected on SJ. This last suggestion leads me to remark on a possible mode of transition between direct and retrograde orbits. The apex of the equilateral triangle is a point of zero force at which the third body may be at rest, although unstably unless the Sun is more than 25 times as massive as Jove Now, it would seem that the body may reach the apex, moving asymptotically with an infinitely small velocity either in the direct or retrograde sense. If this is the case, passage through the apex would afford a mode of transition from the direct to the retrograde.

It is clear that much more remains to be discovered even about these simply periodic orbits, but the present investigation has added something to our knowledge, and it may be hoped that the heavy work which has been undertaken to obtain the foregoing results has proved its justification.

On an Addition to the Theoretical Secular Acceleration of the Moon's Mean Motion. By Ernest W. Brown, M.A., Sc.D., F.R.S.

1. In his paper on the fluctuations of the Moon's mean motion in the *Monthly Notices* for January 1909, Newcomb added $0''{\cdot}27$ to the secular acceleration "on account of the effect due to the combination of the Earth's oblateness with the secular diminution of the obliquity of the ecliptic" (p. 167).

The term is not practically important, but it has raised some interesting points in the theory of secular changes, and has been a subject of discussion, like the principal term in years gone by. There has been no published attempt, beyond a faulty one by the writer, to examine it carefully. As a matter of fact, the possible existence of the term has been known for many years, but the complete theoretical investigation of its value is full of pitfalls, and it has generally been neglected, for one reason or another. I shall try to clear up the matter in this paper; but, before doing so, it may be of interest to give a brief statement of the problem and its history.

2. The figure of the Earth causes certain perturbations of the Moon's motion, which depend, amongst other things, on the angle of inclination of the Moon's orbit to the Earth's equator, and therefore on the inclination of the Earth's orbit to the equator. Now the plane of the Earth's orbit is not quite fixed, but has a slow, secular change of inclination towards the equator, amounting to $47''$ per century. We may therefore expect some secular change in the Moon's motion due to this fact. But the plane of reference

commonly used is not fixed; it is the *actual* plane of the Earth's orbit. The question then arises as to whether the expected secular change will not disappear when we refer the motion to this moving plane. It is further complicated by the mode of approximation usually adopted: we first find the complete motion of the Moon under the assumption that the Earth's orbit lies in a fixed plane, and then we obtain the effects of the motion out of this plane and of the motion of the plane of reference. The term is a second-order effect, depending on the product of the oblateness of the Earth by the motion of the ecliptic.

3. I believe Stockwell was the first to point out and compute this secular change of the second order.* He adopted the simple plan of substituting, without discussion, the variable value of ϵ_1 (the inclination of the ecliptic to the equator) for the constant value in the expression for the variation of the mean motion. This plan is known to give a wrong result if adopted for the principal term of the secular acceleration, and it takes no account of the motion of the plane of reference. In 1899 Mr. Isaac E. Christian † showed by general reasoning that the change of ϵ_1 should produce an addition to the secular acceleration; but again the motion of the plane of reference was not considered. Some two years ago Dr. Hill called my attention to the difficulty, and suggested a more thorough investigation. As a result, I came to the conclusion that no sensible effect would be produced.‡ A year ago Professor Newcomb, obviously unaware at the moment of these publications, wrote to me of his discovery of the term, and expressed his astonishment that it should have been overlooked. Owing to the discussion between us which followed, he withheld the publication of the matter for a time. His insistence on the reality of the term led me to go into the question again, and from different points of view. I finally came to the conclusion that Stockwell's value could be justified, and that my previous conclusion was in error. Professor Newcomb then wrote me that he would publish his note as soon as possible, but I believe he must have omitted to do so beyond the statement quoted at the beginning of this paper.

There seems now to be no reason for further delay in communicating my investigation. I may mention that, according to a brief note of his calculation which he sent me in the course of our correspondence, Newcomb appears to have used substantially the same principle as Stockwell, but he increased the amount by 30 per cent. The numerical result obtained below does not show this increase.

If this secular change exists in the mean motion, there will be corresponding changes in the motions of the perigee and node. These will also be obtained.

4. In the *Monthly Notices* for April 1908 I have given the

* *Theory of the Moon's Motions* (1881), p. 363.
† *Popular Astronomy*, No. 66.
‡ *M.N.*, April 1908, p. 455; *Mem. R.A.S.*, vol. lix. p. 89.

equations of motion of the Moon referred to moving axes in the form

$$\frac{du}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dv}{dt} = -\frac{\partial H}{\partial y}, \quad \frac{dw}{dt} = -\frac{\partial H}{\partial z};$$

$$\frac{dx}{dt} = \frac{\partial H}{\partial u}, \quad \frac{dy}{dt} = \frac{\partial H}{\partial v}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w},$$

$$H = \tfrac{1}{2}(u^2 + v^2 + w^2) - F - R,$$

where R is the disturbing function containing the "gyroscopic" terms due to the motion of the axes, and F is the force function for all the forces acting on the Moon.

Suppose we put $R = 0$. The equations then have the same form as they would have if the axes had been fixed. But F may contain, besides the coordinates x, y, z, constants or variables which are not the same when the axes are fixed. Hence the mere vanishing of R is not sufficient to reduce the problem to one with fixed axes.

5. Consider the various disturbing functions in F. There is first the part due to the Sun. It contains, besides x, y, z, certain quantities supposed to be known functions of t, namely, the solar coordinates. Although the disturbing function is independent of the axes, the functions of the time which have to be substituted for the solar coordinates are not the same for both sets of axes. How do they differ? The coordinates in the plane of $x\,y$ are changed by a quantity which varies as the square of the inclination of the new to the old planes of reference; this may be neglected. We have then the third coordinate z'. For the fixed axes this produces a portion

$$3\frac{m'}{r'^3}\frac{(xx' + yy')zz'}{r'^2},$$

but if we use the true moving ecliptic, $z' = 0$ and the term vanishes.

Now in solving the problem with fixed axes this term was left aside until the end. We see, then, that as it disappears when we refer the motion to moving axes, it is not necessary to compute it at all.

As a matter of fact, I used as the plane of reference the mean instead of the true ecliptic. Hence in R we substitute the mean value of the motion of the ecliptic, and in the above term the periodic variations of the Sun about this mean plane. Hence the term produces no secular change in any case.

Next, the part of F due to the direct actions of the planets. An exactly similar thing happens, but the periodic effects are too small to be sensible.

Third, the portion due to the oblateness of the Earth. The disturbing function contains the angle between the plane of reference (whatever it may be) and the Earth's equator. This angle is changed when the plane of reference is changed. When the latter is fixed it is a constant ϵ_1; when it is movable let the angle be $\epsilon_1 + \epsilon_2 t$. Now the problem was initially solved with $\epsilon_2 = 0$.

We have therefore an addition to the disturbing function depending on $\epsilon_2 t$. This is the portion that was previously omitted, on the wrong assumption that it would be accounted for by R. It is the only part which can give secular changes.

All the portions except the last were computed in detail. We have then only to find the additional effect produced by a change of the constant ϵ_1 into the variable $\epsilon_1 + \epsilon_2 t$.

6. A method for obtaining the effect of such a change is at hand, and is given in chap. xii., sect. (iii) of my new lunar theory.* This requires almost no computation, but the analysis is not simple. What follows is to be considered as an addendum to that theory, and the reader is referred to the Memoir for the notation, etc.

Let δc_i, δw_i be the secular changes of the canonical arbitraries of the solution due to the term $\epsilon_2 t$. It was shown that the changes in the coordinates could be obtained by putting

$$\delta c_i = 0, \quad \delta \epsilon_1 = \epsilon_2 t \qquad . \quad . \quad . \quad . \quad (1)$$

and adding to the arguments the quantities

$$\delta w_i = \int \delta b_i dt \qquad . \quad . \quad . \quad . \quad (2)$$

where all the functions are supposed to be expressed in terms of the canonical constants.

Let us change from c_i to the set of constants n, c_2, c_3 (§ 268), so that c_1 is a function of n, c_2, c_3, ϵ_1. The equations (1), (2) then become

$$\left.\begin{aligned} &\frac{dc_1}{dn}\delta n + \frac{dc_1}{d\epsilon_1}\epsilon_2 t = 0, \ \delta w_1 = \int \delta n \, dt, \\ &\delta c_2 = 0, \quad \delta w_2 = \int \left(\frac{db_2}{dn}\delta n + \frac{db_2}{d\epsilon_1}\epsilon_2 t\right) dt \\ &\delta c_3 = 0, \quad \delta w_3 = \int \left(\frac{db_3}{dn}\delta n + \frac{db_3}{d\epsilon_1}\epsilon_2 t\right) dt \end{aligned}\right\} \quad . \quad . \quad (3)$$

Since we can put $\epsilon_2 = 0$ in the coefficients of δn, $\epsilon_2 t$, the first equation gives δn, and the three equations for δw_i then determine the secular accelerations.

In order to obtain the derivatives with respect to ϵ_1, it is necessary to obtain the changes (denoted below by the symbol Δ) due to the oblateness of the Earth. These changes all contain a factor λ only in its first power, and ϵ_1 enters solely through its presence in λ. Hence

$$\frac{db_2}{d\epsilon_1} = \frac{1}{\lambda}\frac{d\lambda}{d\epsilon_1}\Delta b_2, \quad \frac{db_3}{d\epsilon_1} = \frac{1}{\lambda}\frac{d\lambda}{d\epsilon_1}\Delta b_3;$$

The numerical values of these expressions are known.

To find $dc_1/d\epsilon_1$ we note (chap. xii., equations 9*a*) that c_1 is the

* *Mem. R.A.S.*, vol. lix. pt. i. The remaining references are to portions of this Memoir.

constant term in the development of a certain quadratic function of the rectangular coordinates of the Moon and their derivatives with respect to the arguments. Now the terms due to the oblateness of the Earth contain arguments which are different from any of the arguments arising from other causes. There will therefore be no combinations in the quadratic function giving rise to constant terms; it is to be remembered that the first power of λ is alone retained. Hence the change in c_1 due to the oblateness of the Earth will be entirely due to the constant changes in n, c_2, c_3 due to this cause. In symbols,

$$\Delta c_1 = \frac{dc_1}{dn}\Delta n + \frac{dc_1}{dc_2}\Delta c_2 + \frac{dc_1}{dc_3}\Delta c_3$$

But (§ 270) $\Delta c_2 = 0$, $\Delta c_3 = 0$ and

$$\Delta n = -\frac{1}{a_2\beta}\frac{dR_0}{dn},$$

where R_0 is the constant term in the expansion of the disturbing function for the Earth's oblateness and $dc_1/dn = -a^2\beta$. Hence

$$\Delta c_1 = \frac{dR_0}{dn}, \qquad \frac{dc_1}{d\epsilon_1} = \frac{1}{\lambda}\frac{d\lambda}{d\epsilon_1}\frac{dR_0}{dn}.$$

The form of equations (3) for computation is therefore

$$\delta w_1 = \int \delta n dt, \qquad \delta w_2 = \int\left(\frac{db_i}{dn}\delta n + \frac{1}{\lambda}\frac{d\lambda}{d\epsilon_1}\Delta b_i \cdot \epsilon_2 t\right)dt, \qquad . \quad (4)$$

where $i = 2, 3$ and

$$\delta n = \frac{1}{a^2\beta}\frac{1}{\lambda}\frac{d\lambda}{d\epsilon_1}\frac{dR_0}{dn}\epsilon_2 t \qquad . \quad . \quad . \quad (5)$$

7. The first of equations (4) together with (5) explains why Stockwell obtained the right result by a simple substitution. It can also be seen why a simple substitution will not give a correct result for the main part of the secular acceleration due to the variation of e'. For we cannot neglect powers of e' beyond the first, and consequently the addition to c_1 due to e' contains terms formed from the combinations of periodic terms in the quadratic function mentioned above.

8. In obtaining the numerical values, we may neglect all but the lowest (including zero) powers of e^2, γ^2, that is, of c_2, c_3 present in the various functions.

We have with sufficient approximation (§ 372)

$$R_0 = \frac{1}{4}n'^2a^2\mu_1\left[\left(\frac{a}{r}\right)^3\frac{a^2}{a^2} - 3\left(\frac{a}{r}\right)^5\frac{z^2}{a^2}\right]_0 = \frac{1}{4}n'^2a^2\mu_1,$$

$$n\frac{dR_0}{dn} = \frac{1}{2}n'^2a^2\mu_1, \qquad \frac{d\lambda}{d\epsilon_1} = \frac{d\mu_1}{d\epsilon_1} = -\frac{1}{2}\mu_2.$$

Hence, from equation (5) above,

$$\delta n = -\frac{1}{4\beta}\frac{n'^2}{n}\mu_2 \cdot \epsilon_2 t.$$

If t_c denote the number of centuries, we have (§ 332) $\epsilon_2 t = -47''\cdot 1 t_c$. Also (§ 373) $\mu_1 = \cdot 0000842$, $\mu_2 = \cdot 000242$, and (§ 278) $\beta = \cdot 330$, $n'/n = \cdot 0748$, $n' = 200\pi$. Hence $\delta n = 0''\cdot 40 t_c$ and $\delta w_1 = 0''\cdot 20 t_c^2$, the secular acceleration of the mean motion.

Further (§ 374), $\Delta b_2 = 657''$, $\Delta b_3 = -615''$ and (§ 278) $db_2/dn = -\cdot 0149$, $db_3/dn = \cdot 00373$. Whence $\delta w_2 = +0''\cdot 11 t_c^2$, $\delta w_3 = -0''\cdot 10 t_c^2$, the secular accelerations of the perigee and node.

Adding these to the main portions (§ 370), we obtain for the three secular accelerations

$$+6''\cdot 02, \quad -38''\cdot 2, \quad +6''\cdot 36.$$

New Haven, Conn.:
1909 *October* 20.

On the Plans for New Tables of the Moon's Motion.

By Ernest W. Brown, M.A., Sc.D., F.R.S.

Introduction.

1. The construction of astronomical tables in which the time-element enters is, at the present day, mainly a device to save expense, whether the purpose which they serve is purely scientific, as in the testing of the law of gravitation, or "practical," as in navigation. In the cases where the place of the Sun is required, the problem is simple when once the theoretical expressions for its coordinates have been obtained in terms of the time; these expressions themselves suggest the forms of the tables, and the combinations of terms to be included in any given table. And the tables once made, the labour of obtaining either an ephemeris or a single place is neither troublesome nor extensive. Similar remarks apply to most of the other bodies of the solar system.

The problem in the case of the Moon is very different. Partly owing to its nearness to the Earth, and to the resulting accuracy with which its movements may be traced, and partly owing to the perturbations which it experiences from the action of the Sun and the planets, the terms which should be included are very large in number and complicated in their arguments. They do not naturally fall into a few well-defined groups; there are various modes of expression for the coordinates, each of which has certain advantages and disadvantages for tabulation; the changes from uniform motion are so rapid that its place must be directly computed at very frequent intervals before any interpolations can be made, and some of the constants which materially enter into these

expressions have not been determined with the accuracy which is desired.

There is, therefore, much latitude in the choice of the methods to be used in forming tables. It is true that we have a brilliant set of devices in Hansen's lunar tables, published over fifty years ago, and used as a basis for all the planetary tables which have since been formed. But Hansen's tables, remarkable as they are in their construction, and in the fact that, with slight alterations, they have served for half a century the purpose for which they were made, have become deficient in certain respects. The lunar theory has been extended so as to include many terms not known to Hansen, and greater accuracy is needed, owing to the advance in observational methods. A study of the tables and of the acccumulated experience of those who have used them further suggests that improvements in their form are desirable.

2. Some idea of the new problem may be gathered from the statement that the number of terms to be included is about three times as great as the number included by Hansen. This is partly due to the presence of terms found by his theory, though not included in the tables, but mainly to the new planetary terms. These additional terms, though small in magnitude, are more troublesome to tabulate economically, owing to the difficulty of combining them into a small number of groups suitable for tables.

3. Another factor has entered since Hansen's time. The great development for business purposes of typewriters and computing machines with printing devices suggests possibilities not only in the construction of tables but also in their use. It is quite a matter for consideration whether the tables should not be made with a view to the employment of these mechanical aids. There is also the question whether special machinery, now so easily and accurately constructed, should not be devised in order to abbreviate certain parts of the work of the ephemeris computer. One may perhaps be justified by past progress in looking forward to a time not very distant when, by international co-operation, the regular ephemeris work shall all be performed in a single office, containing the necessary staff of computers and the best machines. Such co-operation would remove the obvious objection to the construction of mechanism devised for a special purpose.

4. In the consideration of the plans outlined in this paper, I have endeavoured to keep in mind these various factors, with due regard to their relative importance. My thanks are due to several expert computers who have freely placed at my disposal their experience in the actual use of lunar and other tables, and particularly to Mr. H. B. Hedrick, who is now engaged in the formation of the tables, and with whom these plans have been frequently discussed. As a result, some general principles in the ease or difficulty of using different kinds of tables have presented themselves. It will be useful to state these briefly in order to avoid continual repetition of the reasons which have led to the adoption of various devices, new or old.

General Principles for the Construction of Tables.

5. The greater part of the labour of computing an ephemeris is the extraction of the numbers, to be afterwards added, from single- and double-entry tables, when the arguments at the starting date have been found. Interpolation is necessary for practically every number, and it is the interpolation that takes time. Hence the first efforts should be directed towards rendering the interpolation easy. This can be done in two ways. First, the difference between the two values between which we interpolate should be as small as possible. Second, this difference, or the variation per unit change of the argument, should be printed; if neither of these is printed, the two values should be in column, so that the necessary mental subtraction is easy. It should not be necessary to consider the variation of this difference: in other words, the second differences should be zero. These rules apply both to the single- and double-entry tables.

Negative quantities should not be used.

The use of trigonometric or logarithmic tables, with the argument changing by varying intervals, should be avoided, since this means a variable interpolating factor. Logarithmic sines and cosines should be avoided, owing to the rapid changes near the zero values. If it is necessary to use them, a zero or very small value of the function can sometimes be avoided by adding a constant greater than unity to the sine or cosine.

In taking a series of values out of a table, the interpolating factor should remain the same, or vary very slowly. The values should be consecutive, and arranged in column rather than in line.

The number of tables should be kept as low as possible. The extent of a single table is of less importance than the number of tables.

Frequent turning over of pages, either in the tables or on the computing sheets, is to be avoided.

Once having entered a table with a given initial argument and interpolating factor, the longer the run without change of argument or factor the better. An occasional slight change of factor is less trouble than the recomputation of an argument.

Where the interests of the ephemeris computer clash with those of the single-place computer, the former are of more importance.

If an adding machine be used for the additions on the computing sheets, the numbers to be added may be placed in line or in column. It is possible to write (or typewrite) from the tables in column, and use an adding machine (as is then necessary) for numbers in a line, or to write in line, and use the adding machine on columns.

Ease in working should not be sacrificed for the sake of symmetry of form or other mathematical property.

6. It is, of course, not always possible to satisfy all these criteria, but they have been kept in mind while the plans were being devised. For example, if no attention had been paid to the

extent of the tables, the interpolations could have been made as small as we wish. I felt, however, that efforts should be directed towards keeping them within a reasonable compass, and that the area occupied should, if possible, not very greatly exceed that of Hansen's tables. I believe that, in spite of the increased number of terms included, this will be found to be the case. Throughout this paper, comparisons with Hansen's forms and tables are continually made. It is to be understood that criticism of Hansen is in no sense implied in these comparisons, but solely a simple means of enabling the reader to form an estimate of the value of the plans set forth below.

The Expressions for the Coordinates.

7. The direct expressions for the longitude on the ecliptic and the parallax, in terms of the time, as set forth in my previous communications,* have been under trial for tabulation. The plans for the latitude are still quite doubtful; they are discussed below.

This direct tabulation has certain exceptions. The terms of very long periods (many years), due to planetary action and to the figure of the Earth, are added to the arguments, and in a few cases become variable factors of the coefficients. The great majority of the planetary terms of short period have been treated in a similar way (see No. 22 below).

8. Hansen expresses the longitude *in the orbit* by means of an auxiliary function z. If the motion had been elliptic, z would have been proportional to t, and the longitude would have been expressed by the well-known formula for elliptic motion, with z as the mean anomaly. The same formula is used for the perturbed motion, and then all the perturbations are added to z. The longitude *on the ecliptic* is obtained by the addition of a function R, "the reduction to the ecliptic," which demands the previous calculation of the latitude; the perturbations of R must also be added.

It has generally been thought that Hansen's mode of expression gives the briefest form. This is possibly true, but it does not necessarily follow that it is the best for tabular purposes. It has certainly the advantage of grouping the larger terms and of diminishing to some extent the number of the smaller terms. But if we have to make a double-entry table containing terms with given arguments, it makes no difference to the use of such a table whether it contains three terms or thirty. The real test is the number of tables required. The deciding factor is not the large terms, but the number of different tables necessary for the small ones. Newcomb's† estimate, that the addition of about 120 numbers would be necessary for the longitude if all known terms were included and the longitude in the orbit were used, is probably

* *Mem. R.A.S.*, vol. lvii. pp. 130–145, and vol. lix. pp. 94–103.

† See the reference in No. 11 below.

not far from the truth. My computation for the solar terms, where the comparison can be made, leads to the conclusion that the number of tables for the two forms would be about the same. The direct tabulation of the longitude *on the ecliptic* avoids two troublesome operations: the entering of the elliptic table with an argument (z) whose interpolating factor requires the use of five-figure logarithms and is different for every entry; and the entering of the table for the reduction to the ecliptic, which requires the previous calculation of the latitude, and in which the successive arguments vary nearly as much as z.

9. The direct tabulation of the latitude (or sine-latitude) of the Moon is nearly as difficult to perform accurately as that of the true longitude A complete set of new arguments is necessary, owing to the presence of odd multiples of F in all terms (the longitude contains only even multiples of this angle), and there are numerous planetary terms which, though small, must be included.

Several experiments have been made with possible transformations of the latitude in order to improve the representation for the purposes of the tables. The form which, so far, appears to promise the best consists in expressing the latitude in the form—

$$(k + \mathrm{C}) \sin (\mathrm{F} + \mathrm{S}) + \mathrm{N}.$$

Here k is a constant; C consists of small terms, which can be put into double-entry tables; S consists (1) of certain large terms which have been tabulated in single-entry tables for the longitude, and whose sum has been found, and (2) of small terms which can be put into double-entry tables. N consists (1) of six large terms put into six single-entry tables, and (2) of small terms involving the multiples 3 and 5 of F. The arguments of the double-entry tables in C, S are the same as some of those used in the longitude, while all the terms in N require new arguments. The double-entry tables in N might be made smaller by tabulating the sine-latitude instead of the latitude. But this device does not eliminate them; it takes out the largest terms, and further requires a table giving the difference between the latitude and sine-latitude, so that little is gained. The question to which attention has mainly to be directed always reduces itself to the tabulation of the small terms. The most troublesome process is the multiplication of $k + \mathrm{C}$ by $\sin (\mathrm{F} + \mathrm{S})$.

The greatest advantage gained is in the number of double-entry tables. About half as many more would be required for the same accuracy if the original expression for the latitude were used.

The transformation of the latitude into the new form has already been completed, preliminary trials appearing sufficiently promising to justify the undertaking; the methods and results will form the subject of a separate communication to the Society. From a theoretical point of view, the new form must be regarded as arbitrary, since no theory has given the latitude in this form. Partial justification may, however, be made. If we had used the

method of the variations of the node and inclination, the latitude would have come into the form—

$$\text{Sin lat.} = (k + \text{C}) \sin (\text{F} + \text{S}).$$

Then C would contain the perturbations of the sine-inclination and S those of the elliptic longitude less the longitude of the node. But of course the terms included in N would have been present in a different form in C, S.

10. Hansen really keeps his inclination constant while using the true longitude in F + S. He puts the sine-latitude into the form $\text{K} \sin\phi + s$, where K is a constant, ϕ is the longitude in the orbit less the motion of the node, and s contains the remainder of the perturbations—tabulated into single- and double-entry tables. The latitude is then deduced by a table giving the difference between the angle and its sine. This is the form which arises naturally with his theory, and it has the advantage of avoiding the necessity for tabulating the large elliptic terms which, when the longitude is known, may be included in ϕ. But all the arguments in s are new. The disadvantage of the form arises with the terms dependent on the solar eccentricity, which cannot be chiefly contained in ϕ.

11. Newcomb's long experience in the formation of tables had naturally led him to consider improvements in the form and arrangement of the lunar tables, and, at the special request of the writer, these were embodied in a paper which was published in the *Monthly Notices* for June 1908. In this paper he suggested that the longitude in the orbit should be tabulated, together with the reduction to the ecliptic, somewhat in Hansen's manner. This is based on the idea that since the solar perturbations of the latitude apparently require fewer terms when expressed in Hansen's form, they will therefore require fewer tables. My own comparison does not bear out this conclusion; at any rate, the difference is so small when we attempt to include many terms not tabulated by Hansen that it need scarcely be considered.

Units and Degree of Accuracy.

12. The unit of time is the mean solar day, and the arguments are expressed in days and decimal parts of a day.

The coefficients of the periodic terms are expressed in seconds of arc. The unit of the tables for the longitude and latitude is 0″·01, and for the parallax 0″·001. After the coordinates have been found to this degree of accuracy, the last place of decimals in each case is cut off.

13. The actual tables are being computed* to 0″·001 and 0″·0001 respectively; it is intended that the last figure shall be cut off in the printing. It is an open question whether a somewhat greater accuracy in the separate tables for the longitude and

* Only the single-entry terms in the longitude are at present in the computer's hands.

latitude is desirable, say to 0″·005. This may be obtained in two ways. A dot may be placed after the last figure when the figure cut off is 0, 1, 2, 3, or 4. Figure 2, for example, with a dot, would then be read 2·2, and without a dot, 1·7 or 1·8. The dot would only be used in connection with the interpolation, and would not appear on the computing sheets. The other method is to multiply the tables, before printing, by 2, and make a final division by 2 after the places of the Moon have been obtained. The former method seems to be preferable, on the whole; the computer need pay no attention to the dot unless instructed to do so, and in any case he soon becomes used to a device common to logarithmic and other tables. Still greater accuracy could be obtained with another sign, but it would be of very little additional value unless used on the computing sheets, and this is not desirable. The second method is open to the objection that it doubles the interpolating factors, and in a good many cases adds another figure to the tabulated values; it also compels the computer to a degree of accuracy which, after all, may not be necessary.

14. The largest interval for which final interpolation to any date is possible is about half a day, and this is determined as much by some of the smaller terms with very short periods as by the principal elliptic term with its period of $27\frac{1}{2}$ days. The interval corresponds nearly with the final degree of accuracy required. Although increased accuracy of observation in the future may sometime require that the interval be diminished, the present prospect of a demand for a degree of accuracy beyond that of the new tables is so small that it does not seem necessary to prepare for it. It may be stated that, when needed, the demand can be met by additional tables, for which the expense and trouble of preparation will not be great.

15. In his paper referred to above, Newcomb considered in some detail the accumulated errors of the half-day ephemeris. He suggested that the fifth or sixth differences should be smoothed so as to distribute the errors amongst the adjacent values. Under proper precautions this may be done. It is to be remembered, however, that the magnitude of the sixth differences resulting from theory will be of the same order as the errors resulting from the use of decimals. It would be necessary, for correction, to proceed to seventh or eighth differences, when a definite rule as to the greatest difference to be allowed might be formulated. In fact, some of the periods are so short that the theoretical differences diminish very slowly. The ratio of the $2n$th difference, in the case of each harmonic term, to the term is $4^n \sin^{2n} 2\pi/2p$, where p is the number of half-days in the period. The differences therefore increase with n unless the period is greater than 3 days. A few small terms come near this limit, rendering any deduction drawn from the differences of a high order somewhat doubtful. The fact is that the half-day is not too far away from the maximum interval which will work for differencing purposes.

16. A further point which has to be considered at the outset

is the number of lines to a page. In the synodic month there are 59 half-days, and in most cases (see Nos. 39, 49) the reading for consecutive half-days will be vertical instead of horizontal. The basis was therefore about 65 lines to the page, with a possible extension to 75 lines by the use of a different font. The length of the page is then practically the same as that of Hansen's page. The width of a page is a matter which can be left until the tables are nearly ready for printing.

The above considerations are those which have been mainly responsible for the forms of the tables suggested in this paper.

Plans for Tabulating the Longitude.

17. The expression for the longitude consists of sums of harmonic terms whose arguments are sums of multiples of a few fundamental arguments. The latter are—

D, the difference between the mean motions of the Moon and Sun.
l, the Moon's mean anomaly.
$2F$, double the mean argument of the latitude.
l', the Sun's mean anomaly.
w_3, the longitude of the Moon's node.
Q, V, M, J, S, the mean longitudes of Mercury, Venus, Mars, Jupiter, and Saturn.

The so-called "solar" terms depend on the first four arguments only; the terms due to the oblateness of the Earth depend on the first five; and the planetary terms depend on all the arguments. Certain small terms contain slight differences from these arguments, due to secular changes, and the planetary terms contain constant angles which in general are different for each term.

18. The usual plan is first to take out the terms with large coefficients and to place them in single-entry tables. The remaining terms are then put into double-entry tables. Let α, β be any two of the above arguments, or any two given combinations of them. A double-entry table is formed of all the terms containing the combinations $i\alpha \pm j\beta$, where i, j are integers. For economical use, one of the arguments, α (or its multiples), must be the same for a large number of tables. The second argument may therefore consist of many combinations of the other arguments, and each combination, with its multiples, requires a double-entry table.

The first thing to settle is the division between the terms put into single- or double-entry tables. This depends on the size of the coefficient and the period of the argument. The chief factor in the decision is the interpolation, on the computing sheets for the double-entry tables, from the tabular argument α to the value of α at the required date. This interpolation is in general to require the use of second differences only, and never more than third differences. The division will be shown by the classification given

below of the terms put into single-entry tables (No. 45), and of those which have to be inserted in double-entry tables (No. 51).

19. The use of a double-entry table thus demands a double interpolation, which is done in two stages. In entering the tables, the argument common to all the tables has the exact values for which it is tabulated, so that, in extracting numbers from the tables, the interpolations are made for the second argument only. As soon as all the numbers from the double-entry tables have been written down and added, a final interpolation to the required values of the argument common to all is made. When ephemeris computations are required, the period of the common argument therefore determines the number of entries to be made on the computing sheets at one time.

In the tabulation of the solar terms, there are two arguments which naturally present themselves for the common argument, owing to the frequency of their occurrence,—l, the mean anomaly, used by Hansen, and D, the difference of the mean motions of the Moon and Sun. It is possible that the mean anomaly is the more convenient with Hansen's method of expressing the coordinates of the Moon. But if the Moon's longitude be found directly, the advantage seems to be with D. In the first place, the number of the terms which are independent of l, when the four angles D, l, F, l' are used, is quite large, while practically every set of terms which contains any combination of l, F, l' will also contain D. A second advantage arises from the fact that D is known with practical certainty, and its secular change is only one-sixth that of l. Further, the period of D being $29\frac{1}{2}$ days and that of l being $27\frac{1}{2}$ days, the ephemeris computer will in general have to calculate one monthly block less in the year; of course each block will be four half-days longer, but the labour due to having four extra half-days in the month is considerably less than that of computing an extra block, with the accompanying changes of arguments and interpolating factors. For these reasons D has been considered the best common argument of the solar terms in the double-entry tables.

20. The terms due to planetary action and containing the planetary arguments are divided for tabulation into three classes.

The terms of very long periods (many years) are treated with the terms added to the arguments in No. 28.

21. Next come the terms which are independent of the special lunar arguments l, D, F, and which therefore depend on the Sun's longitude and the longitude of a planet. These are similar to the perturbations of a planet, and are tabulated in much the same way. There are five planets giving sensible terms—Venus, Mars, Jupiter, Saturn, and Mercury. The terms due to Saturn and Mercury are, however, so few and small that it may be better to place them amongst the remainder terms (see No. 23). The synodic periods of Venus and Mars are roughly $1\frac{1}{2}$ and 2 years respectively, and the period of Jupiter is 12 years. Further, slow convergence takes place chiefly along multiples of the two synodic periods. Hence it

will be best to take T, the Earth's mean longitude (or l', its mean anomaly) as the first argument, and the three periods named as second arguments of the double-entry tables. The interval for T will probably be ten days, but in other respects the tables are similar to the other double-entry tables described above. When the sums of the entries at intervals of ten days have been formed, interpolation to halves with second differences, and then to tenths with first differences, will give the half-daily values.

22. The last class consists of the terms containing both the planetary and lunar arguments. By far the greater number of these terms in longitude contain only the first multiples of the three lunar arguments l, $2D - l$, $2D$, and this fact suggests that the terms be considered as corrections to the arguments and coefficients of the three principal terms with these arguments. (It is interesting to note that this is not the way that many of these terms present themselves when obtained by the variation of arbitrary constants.)

Consider the principal elliptic term $k \sin l = k \sin (l_0 + l_1)$, where l_1 is so small* that its square may be neglected, and l_0 is the argument nearest to l in the single-entry table for this term. We then have $\sin l = \sin l_0 + l_1 \cos l_0$ and l_1 is the interpolating factor. This factor remains unchanged for at least ten days, and has only slow variations, due to the secular change of l; to certain very long-period terms added to l (No. 28); and to the adjustment mentioned in No. 41. In the elliptic table we have the values of $k \sin l_0$, and $k \cos l_0$, since the variation of $k \sin l_0$ per $0^d{\cdot}001$ is given.

Let us express the planetary terms containing l in the form

$$k_1 \sin l + k_2 \cos l = \left(k \sin l\right)\frac{k_1}{k} + \left(k' \cos l\right)\frac{k_2}{k'}.$$

Then k_1, k_2 will consist of terms containing the planetary mean motions only, and they therefore may be put into double-entry tables like those for the terms independent of the lunar arguments. Further, k_1, k_2 are so small that we may neglect their squares and higher powers, and replace l by l_0. Hence, having obtained the values of k_1/k from the double-entry tables and added them, we have to multiply the sum by the value of $k \sin l_0$, as found from its table: since k_1/k will not exceed three significant figures, and changes slowly, this multiplication may be done by Crelle. The portion $\frac{k_2}{k'}$ is found in the same way, but before multiplication by Crelle the interpolating factor l_1 (with its changes) is added. To each of these sums are also added before multiplication certain of the remainder terms considered in No. 24.

The terms depending on $2D - l$ and $2D$ are treated in the same way. Amongst them may be similarly included the small secular variations of the coefficients, and it will be seen below that the interpolating factors and some other variations may also be included. Thus the method of abbreviating the tabulation of the

* Less than ·001 of a day.

planetary terms of short period also permits of the inclusion of several other classes of terms which would otherwise require separate treatment.

23. After all the selections for the tables above described have been made, there still remains a large number of terms, about 150, which it is desired not to neglect. No one of these terms has a coefficient so great as 0″·100, and all below 0″·003 are neglected. The periods approximate to a year and sub-multiples of a year, and to a month and sub-multiples of a month. If these were put into the large number of single- and double-entry tables which would be necessary, though the trouble of constructing such tables is insignificant, the labour forced on the ephemeris computer would be quite large in comparison with the other portions of his work. The only other method which appears to be available consists in finding directly the sums of these terms for a sufficient number of dates in the past and future, and printing these sums in a separate table. A variation of this method, which is only useful to the ephemeris computer, consists in forming these sums by some mechanical method.

The known degree of accuracy of past observations points to a century as the extreme limit in the past to which these remainder terms of very short period need be computed. Whatever method be used, it is hoped that the sums may be found for this period and printed. As to the future, if the mechanical method turns out to be successful, it would only be necessary to have the special computing machine in one of the offices which compute the ephemeris of the Moon ; the values for each year could then be run off as they were needed, and, if necessary, communicated to the other offices.

24. Instead of attempting to put all the remainder terms into one sum—a too great task—they are divided into four classes.

First, the terms which have either such small coefficients or such long periods that they need be tabulated only once in every ten days. A table of these for 200 years containing about 7300 values could be put into 5 pages or less.

Second, the terms whose periods approximate to a month. Each term is expressed in the form $a \cos \lambda \sin l + a \sin \lambda \cos l$, where l is the mean anomaly and λ is a long-period argument. Tables for $\Sigma a \cos \lambda$, $\Sigma a \sin \lambda$, precisely similar to the table just described, are formed with ten-day intervals for 200 years. The sums are added to the planetary terms described in No. 22, so that no new multiplications by $\sin l$ and $\cos l$ have to be performed.

Third, the terms whose periods approximate to half a month. These are expressed in the form $a \cos \lambda \sin 2D + a \sin \lambda \cos 2D$, where 2D is the argument of the variation. The treatment is the same, and the $\Sigma a \cos \lambda$, $\Sigma a \sin \lambda$ are added to the planetary terms which have 2D in their arguments.

Finally, there are about 50 terms of shorter periods, ranging from 1/3 to 1/9 of a month. These, if computed for every half-day for a century, would occupy a table with about 73,000 entries. No

entry would have more than two figures, and they could be arranged so as to fill not more than 20 pages.

Forty-five or fifty pages seems a disproportionate amount of space to devote to these small terms, but it will be remembered that these are not tables for computation, but results of computation, of the kind done each year for the ephemeris. By expressing the tabulation as an average of half a page per annum, we obtain a more just idea of the method. At the same time, some improvement on the plan is desirable.

25. The possibility of treating the terms in the last class economically rests very largely on the successful construction of the special computing machine already referred to. I hope to describe it on a future occasion. So far, its general plan has been under consideration and discussion during the past twelve months, and the parts likely to cause difficulty have been tested by actual construction. The problem is similar to that of finding the tides at a given port when the constants of the port are known. It is more simple, in that the changes are much slower than those of the tidal curve, but the number of terms to be included is greater. The cost of the tidal machines devised up to the present time renders it advisable to attempt the construction of one for the tabulation of these terms on different lines.

Tabulation of the Arguments.

26. The method for tabulating the arguments, with their mean motions and secular changes, is in general the same as in other tables. If the year be used as the unit, the values of the arguments are given at the beginning of every century likely to be used; a further table of the additional parts for the beginning of each year in any century is added. With these motions may be included a few small terms of periods many centuries long, and with variations which may be neglected during a single century. The mean longitude is tabulated in the same way, and is included in the discussion which follows.

27. A question raised by Newcomb in his paper referred to above is whether the year is the best unit. Hansen used it, and it certainly makes the deduction of the arguments from the calendar very easy. Peirce, in his tables of the Moon, and Oppolzer, in his *Canon der Finsternisse*, adopt 1000 days; another table is then necessary to reduce the calendar date to multiples of 1000 days from the epoch. Owing to the larger numbers of multiples of terms of short period which would have to be frequently subtracted, Newcomb inclines to 500 days. The matter is not very important for ephemeris work, and in any case it can be left open for the present, since the formation of tables for the arguments is the final stage.

28. It is convenient to tabulate a number of terms of very long period (many years) as additions to the arguments, though

this is not a necessary form. With one serious exception, the quantity which is alone affected by these terms is the mean longitude, but this mean longitude is present in l, D, F, so that several of the arguments need to have the addition. The exception involves terms (chiefly due to the figure of the Earth) which have a period of about 19 years. The quantities affected are the node, the inclination, the mean motion, and the perigee. The variations of the node and inclination are of the same order, and affect the latitude to about the same degree that the variation of the mean motion affects the longitude; the variation of the perigee has but small influence. The trouble in the longitude is due to the effects on all the larger terms, and also on the terms containing the argument 2F. The only other term with influence on many arguments is the large Venus inequality, and its effect is almost entirely through the mean motion.

29. Outside of these terms we have quite a number of small long-period terms, chiefly affecting the mean longitude, and through it the three or four largest periodic terms. The arguments of these long-period terms are nearly all quite independent of one another, so that, instead of attempting to put them into a number of separate tables, it would appear to be best to calculate them for a number of dates backwards and forwards from the present time. Semi-annual values would be sufficiently frequent, and 300 years back will be ample for all observational records; the same length of time forwards would provide for future needs. There would then be 1200 values for each of the four or five arguments affected, and some four pages or less would suffice.

30. The big Venus inequality requires different treatment. It is tabulated for the mean longitude by a single-entry table in the ordinary way. But it affects quite a number of other arguments; the effect on any argument bears a constant ratio to that on the mean longitude. Hence the simplest plan is to give a table of these constant factors, one for each argument. The computer first takes out the mean longitude value for the date or dates required; he then has only to multiply the values by the constant factors to get the values for the other arguments. The period of the term is 273 years, so that only a very few values per annum are required. Crelle's multiplication table can be used.

31. The same treatment can be adopted for the 19-year terms. The intervals must be rather shorter, perhaps once a month in all arguments except the mean longitude; the interpolation to half-days is performed on the sum of this with the other long-period terms. Besides the effect on the arguments, we have a similar effect on the inclination. A separate table is given for this. The value is used as a small factor of the sum of the terms containing the argument 2F.

32. The secular variation of the solar eccentricity must be included in the coefficients of several terms: its ratio to the solar eccentricity is approximately $\frac{t-1900}{40{,}000}$, where t is the date in

years. This is most easily treated without a table by using this quantity as a factor of the sums of the terms affected after they have been entered on the computing sheets.

33. All the terms present in the expression for the longitude on the ecliptic, and their arguments, are thus accounted for. The methods are not entirely satisfactory, and it is hoped that improvements may be made before the time when the final tables of these quantities must be computed. It is fortunate that the tables which require most labour are also those which are likely to need the least change either in form or values, namely, the solar terms, and they can therefore be undertaken first. Next come the numerous planetary terms, then the long-period variations of the arguments and coefficients, and finally the arguments themselves.

A new form has been devised for the single-entry tables. The double-entry tables have also had certain changes made from the forms given by Hansen, and followed by others. For the ephemeris computer the changes are important, and they will be described in some detail in the following pages.

Form of the Single-Entry Tables.

34. The place of the Moon is to be computed for every half-day, this being the longest interval for which interpolation is possible with the accuracy required. To perform this interpolation requires the formation of the differences of a high order (sixth or seventh), and it can therefore only be economically used in the final places, and not on the separate tables. The arguments of the separate tables may be anything whatever; and to obtain the values corresponding to a given date, so that the interpolation from any table shall only involve the use of first differences, it is necessary to have the intervals of the arguments in the tables much smaller than half a day. The table must therefore take the form of a double entry table in which one argument is half a day and the other (for interpolation purposes) a small fraction of a day. Further, as every quantity to be tabulated in this way is periodic, in order to avoid repetition the arguments tabulated should not exceed the period of the term.

35. Hansen has arranged each single-entry table in a block, in which the interval in one direction (horizontally) is a half-day and in the other direction (vertically) the hundredth part of a day. Thus for any given initial argument, say 5·7327 days, the computer chooses the column headed 5·5 days, and the lines in that column corresponding to ·23 and ·24, and interpolates for the interval ·0027 in order to get the value corresponding to 5·7327. For the succeeding half-days, namely, 6·2327, 6·7327, . . . , the computer keeps along the same two lines, using the same interval ·0027, and merely changes from one column to the next. When he gets to the end of the table, where the argument exceeds the period of the term, it is necessary to subtract the period and

start afresh with a new argument. Hansen wishes to be able to go for a whole month without such a change; the horizontal extent of his table must therefore contain a number of columns equal to the number of half-days in the period, plus the number of half-days in the month. This amounts to a tabulation for intervals of which no use can be made in the division of the argument.

For example, suppose the period is 13·357 days and the month is 27·551 days. As we may have to start with any argument less than 13·357, in order to continue for a month it is necessary to tabulate for $13{\cdot}357 + 27{\cdot}551 = 40{\cdot}908$ days, that is, we require 82 columns. We thus have over three periods of the term actually tabulated, but for the interpolation we get no benefit from this fact, since we start with arguments present in the first period only. It ought to be possible to either diminish the table to one-third of its extent, or diminish the interval for interpolation to 1/300th of a day, in order to make the table of full value.

36. Mr. P. H. Cowell* suggested that the terms should be tabulated for successive hours through several periods, an extra table being provided to give the value of an argument greater than a period in terms of one less than a period, so that the interpolation is made as small as is consistent with the number of entries in the table. The plan does not, however, seem feasible, chiefly because the hourly computation of the place of the Moon is too expensive for ephemeris work. For obtaining an isolated place, the plan would probably be economical. The reading of his paper, however, was the starting-point in the plan which is worked out below.

37. *The new plan.*—The central idea is the tabulation for successive half-days for a number of periods so chosen as to divide up the argument to the required degree of smallness, while arguments less than a period are alone inserted. Its special features are: (1) the argument for a given date being given, the values for all successive half-days from that date can be obtained without change (or with very slight changes at long intervals) of the interpolating factor. (2) For a given division of the argument the extent of the table is $\frac{1}{n+1}$ that of Hansen, n being the number of times the period is contained in a month; or (3) in the same space the argument may be divided into $n+1$ times as many parts as Hansen divides it. (4) Using (3) rather than (2), it has been found that the values for successive half-days can be arranged vertically instead of horizontally. (5) The differences for interpolation can always be printed without much increasing the space required.

38. The plan, though somewhat troublesome to explain, is sufficiently simple in actual use. I shall illustrate it by an example, the tabulation of the term with argument 2F, coefficient 412″, and period 13·60611003 days.

We first find the successive convergents of the decimal fraction

* *The Observatory*, vol. xxvi. pp. 287–290.

27·21222006, this being the ratio of the period to half a day. They are 27 plus

$$\frac{1}{4},\quad \frac{1}{5},\quad \frac{3}{14},\quad \frac{7}{33},\quad \frac{59}{278},\quad \frac{66}{311}.$$

If any one of the last three of these convergents be used instead of the actual period, the errors of the arguments after the lapse of 33, 278, and 311 periods will be respectively

$$0^d{\cdot}0016,\quad 0^d{\cdot}0014,\quad 0^d{\cdot}00023.$$

Suppose the last convergent written be chosen. After 311 successive periods, the function and the argument return to their original values. There is an error, however, owing to the period not being the actual period;* but after 730 sucessive half-days (a year), the error of the term will be less than $0''{\cdot}004$, a sufficient degree of accuracy.

Next, the use of 311 periods means that the half-day has been divided into 311 parts, and that a whole period has been divided into 8463 parts, each equal to $0^d{\cdot}001608$. We have therefore to compute the term for one period at intervals of $0^d{\cdot}001608$, and we know that if we start with $0^d{\cdot}0$ we shall have in the series the exact values for 8463 successive half-days, after which the function and its argument return to their original values, and we can proceed indefinitely.

39. The rest of the work is therefore simply the arrangement of the values in such a way that the ephemeris computer may do three things easily: (1) obtain the value corresponding to any initial argument; (2) follow through successive half-days; (3) interpolate when the given argument is not exactly found in the table.

The columns are headed with the arguments $0^d{\cdot}0000$, $0^d{\cdot}001608$, $0^d{\cdot}003216$ . . . . and so on by multiples up to $0{\cdot}5000$: there are 311 of these columns. The arguments at the side are $0^d{\cdot}0$, $0^d{\cdot}5$, $1^d{\cdot}0$, up to $13^d{\cdot}5$, so that the side arguments are the whole number of half-days, and the top arguments the fractional parts. The corresponding values of the term are inserted in the body of the table.

The columns are numbered 1, 2, 3, . . . 311 at the head, and there are numbers placed also at the foot to indicate the next column to be followed when the foot of any column is reached. Suppose he starts with the argument $0{\cdot}000000$. He can find the values for successive half-days up to $13{\cdot}5$ by reading down the first column. At the foot of that column he finds the number *a*, which indicates that the value for the next half-day is the first number in column *a*. That column is followed down to its foot in the same way, where the number *b* is found, and he therefore next follows column *b*. The process can be continued through the whole 311 columns, and at the foot of the last column used he finds the number 1, indicating that he starts again on column 1.

* The idea of using an adjusted period instead of the real period, in order to make the table *completely* re-entrant, was suggested to me by Mr. Hedrick. It has the further advantage of making all the intervals equal.

If the starting argument is not exactly $0^d{\cdot}000000$ but say $11^d{\cdot}502311$, he starts by choosing the line $11^d{\cdot}5$ and the column $0^d{\cdot}001608$, and interpolates for $0^d{\cdot}000703$, the difference for $0^d{\cdot}001$ being given in each case. The same process is followed with this interpolating factor, which remains the same throughout.

40. The table is reduced to half its extent if there is a single term, or if all the arguments of the included terms are odd multiples of one of them (as in this example). Half of the decimal headings are above and the other half below. There is a slight change in the arrangement which compels the computer to read up instead of down after a certain number of columns have been used, but a second change is not needed until all the columns have been used.

41. In the case of a few large terms it is not convenient to use such a large number of periods that there is no correction to the argument during a year's run. The correction is put in the form of a small addition to the interpolating factor, and it will be given by a note at the foot of each page of the table.

42. There are also a few periods for which the convergents run badly. For example, with the argument 2D we have the successive convergents $29\frac{26}{49}$, $29\frac{425}{801}$. The use of the latter would make the table too large, while the former does not divide the half-day into a sufficient number of parts. In this case I divide the half-day into 6×49 parts, so that the interval is $\frac{1}{588}$ day $= 0{\cdot}0017007$. The interpolation factor will require a slight correction from time to time. The term in longitude is $k \sin 2D$. Now $\sin 2D = \cos(2D + \frac{1}{4}$ period), and I choose the initial arguments of the table to be the values of 2D, for which

$$2D + \tfrac{1}{4} \text{ period} = (n - \tfrac{1}{2})\ (0^d{\cdot}0017007), \qquad n = 1, 2, 3, 4, 5, 6.$$

Suppose we have an initial argument which requires us to make use of the columns for which $n = 1$, reading down. We follow these columns through until we reach the foot of that column which requires a reversal of the direction of reading from down to up. The columns for which $n = 6$ are then followed upwards until we reach the top of the column where we have to reverse again from up to down, and the change is found to lead us back to the columns for which $n = 1$. Hence the columns for which $n = 1$ (down) and $n = 6$ (up) form a complete circuit. Similarly, those for which $n = 6$ (down) and $n = 1$ (up) also form a circuit. An exactly similar thing happens with the pairs $n = 2$, 5 and $n = 3$, 4. It should be noted that the ephemeris computer has no need to take any account of these facts: he simply proceeds to the column shown by the number at the foot when reading down, or conversely when reading up.

43. It has been stated above that for interpolation purposes the variations of each value for $0^d{\cdot}001$ will be printed. It should be noted that the division of the argument has been made so small that the change in these variations from one column to the next is very slow; in fact, one column giving these variations will suffice

for several columns of values of the function. Further, the variations, with three exceptions, never exceed 40 units, so that the interpolation will be very simple. The exceptions are in those terms for which the interpolating factor is added to the planetary terms (see No. 22), namely, those with arguments l, $2D - l$, $2D$. Even here the interpolating factor does not exceed three figures.

44. The space occupied by the single-entry tables in this form exceeds that of Hansen's tables for the corresponding terms very slightly. Although the arguments are much more minutely divided, a saving is effected, not only by using for interpolation purposes every period tabulated, but also by the possibility of being able to use one column of variations for several columns of values, as is usual in logarithm tables.

A further advantage arises from the fact that all the short periods are approximately one, one-half, one-third or one-quarter of a month long, so that the columns are approximately 60, 30, 20, or 15 lines long respectively. The pages, therefore, have respectively one, two, or three blocks of columns on them, and the tables can follow one another without loss of space.

45. The following table shows those terms, arising from the action of the Sun, which have been placed in single-entry tables. The first column gives the arguments; in the argument iD are included all multiples of D except the multiple 2. This table is the only one which is not reversible.

The second column gives the coefficient of the largest term, and the third column gives the period.

In the fourth column is placed the number of periods actually included in the table. When the convergents are unsuitable, the subdivisions (see No. 42) are shown by the factor given.

The fifth column shows the interval of interpolation; it is a half-day divided by the numbers in the fourth column. The last column gives the maximum difference of two consecutive values in the table; the amount added or subtracted on interpolating, therefore, never exceeds one-half of these differences.

The numbers, except those in the fourth column, are approximate.

45. *Single-Entry Solar Terms.*

Argument.	Coef.	Period.	No. of Periods.	Interval.	Max. Diff.
	″	d		d	″
l, $3l$, $5l$	22640	27·55	55×6	0·0015	7·8
$2l$, $6l$	769	13·78	110×4	·0011	·40
$2D - l$, $6D - 3l$	4586	31·81	335	·0015	1·3
$4D - 2l$	31	15·91	117	·0043	·06
$2D$	2370	14·77	49×6	·0017	1·8
iD	125	29·53	49×2	·0051	·20
$2D - l - l'$	206	34·85	258	·0019	·08

Single-Entry Solar Terms—continued.

Argument.	Coef.	Period.	No. of Periods.	Interval.	Max. Diff.
	″	d		d	″
$l-l'$	148	29·80	71 × 2	·0035	·11
$l+l'$	110	25·62	189	·0027	·09
$2F-l$	40	26·88	152	·0033	·03
$2D-l+l'$	29	29·26	207	·0024	·02
$2F$	412	13·61	311	·0016	·31
$2D-l'$	165	15·39	599	·0008	·06
$2D+l'$	25	14·19	167	·0030	·04
$2D+l$	192	9·61	277	·0018	·23
$2F+l$	45	9·11	189	·0027	·09
$4D-l$	38	10·08	396	·0013	·03
$2D+l-l'$	15	9·87	178	·0028	·03
$2D+2l$	14	7·13	299	·0017	·03
l'	668	365·26	25	·020	·25
$2l-2D$	212	205·89	14	·036	·23
$2F-2D$	55	173·31	21	·024	·05

The number of columns in a table is half the number of periods except in the case of iD, where it is the whole number.

The estimate of space with a page like that of Hansen is 90 to 100 pages for the 22 tables.

The Double-Entry Tables.

46. There is one feature of Hansen's double-entry tables which has always caused trouble and discussion. Although, both by precept and example, he obviously intends to arrange for a computation of the Moon's place every half-day, he has given successive columns in these tables for every quarter-day, so that the ephemeris computer has to take alternate columns. The reason for thus doubling the size of the tables was a simple one. The computer chose the quarter-day nearest to the argument which was needed, and thus in the interpolation of the sums of the double-entry tables his maximum interpolating factor was $\frac{1}{4}$: this maximum factor would have been $\frac{1}{2}$ if Hansen had omitted the tabulation of the quarter-days. A little trouble is saved in this final interpolation, but it is more than compensated by the increased care necessary in dealing with alternate as against consecutive columns. The latter would have been avoided if the even and odd quarter-days had been put into separate tables. Probably the sole advantage of Hansen's arrangement is gained in the computation of an isolated place of the Moon. If the odd quarter-days be omitted, it is true that one more place must be computed for interpolation purposes when finding an isolated place, but this increase of

work will be fully counterbalanced in ephemeris computation by the improvements suggested below.

There is only one other difference in actual form from Hansen's tables, namely, the reading for consecutive half-days vertically instead of horizontally. Hence it will be simplest to describe Hansen's double-entry tables and note the modifications.

47. Hansen divides his second argument into an arbitrary number of parts, always a factor of 360 or 720, and so chosen as to divide up the argument into a number of parts sufficient for interpolation with first differences. The advantage of using a whole number of degrees or half-degrees lies almost entirely in the ease of making the tables. I propose to make use of the arbitrary division so as to simplify the work of the ephemeris computer.

The month always begins at the time when D = 0. Hence, in proceeding from one month to the next, the second argument has to be changed by an amount

$$\frac{\alpha}{n - n'} \mathrm{P},$$

where α is the mean motion of the second argument, P the number of parts into which it is divided, and $n - n'$ is the mean motion of D. If, after the addition, the argument exceeds the period, then the whole number of parts must be subtracted. As this must be done for every month and for every argument, there will be advantage if the additive part be a whole number. This can be achieved approximately by using one of the convergents to the decimal fraction $\alpha/(n - n')$: if the denominator be taken for the number of parts, the numerator will be the monthly addition.

This convergent must be close enough to the true ratio of the periods to prevent any accumulated error of the argument in a year, and the number of parts should not be greater than is necessary, lest the tables be unduly expanded. Many of the convergents do not satisfy these two tests simultaneously. The difficulty is avoided by finding the convergents to $10\alpha/(n - n')$, or, if that is not close enough, to $100\alpha/(n - n')$. The monthly addition is then an integer or an *exact* number with one or two decimal places, while the number of parts is an integer.

The method is essentially the same as that adopted for the single-entry terms, but there the addition at the end of a period to get the next half-day was always a whole number, denoted, symbolically, by the number of the column which is to be followed at the close of any one column.

48. The second argument, by subtraction of the proper number of multiples of D, can always be made of longer period than D.

The extent of a table can be reduced, as in Hansen, whenever odd multiples of the second argument are alone present. This is the case with two-thirds of the double-entry tables.

49. The arrangement so that consecutive half-days may be read vertically instead of horizontally requires that the differences be printed to avoid the difficulty of mentally forming the differ-

ence between two consecutive members in the same horizontal line. This will require an increase of about 50 per cent. of space, but the omission of the odd quarter-days will more than provide for it.

A further advantage of this arrangement is that every column of every table is the same length—about 63 lines; the number of columns will alone vary from table to table.

50. The forms for the double-entry planetary tables are similar; but since the common argument is the year, and since the argument for the beginning of the year alters very slowly, it is unnecessary to use the convergents instead of the actual periods. The most suitable number of divisions of the second argument will suffice.

As before, it will be convenient to have the values for consecutive ten-day intervals in column, and to give the variations. With this interval each column will contain about forty lines, a somewhat inconvenient length for economy of space.

51. The following table exhibits the principal facts concerning the solar terms which are intended to be included in the double-entry tables.

The first column contains the second argument; the second, the greatest coefficient amongst the included terms; the third, the number of terms having coefficients greater than 0″·002 to be included; the fourth column shows the number of parts into which the argument is divided; and the last, the exact addition to the argument necessary to obtain its value at the beginning of the succeeding month.

The star indicates that the table is not reversible. In these cases the number of columns is equal to the number of parts; in all the other tables it is equal to half the number of parts.

Double-Entry Solar Terms.

Second Argument.	Max. Coef.	No. of Terms.	No. of Parts.	Monthly Addition.
	″			
$l - D$	19	31	251*	18
l'	18	17	161*	13
$l + l' - D$	5	13	78*	11·9
$l - l' - D$	1	13	58*	·53
$2l + l' - 2D$	9	12	103	23·1
$2l - l' - 2D$	10	10	155	9·7
$l + 2l' - D$	8	7	129	30·11
$l - 2l' - D$	3	6	100	9
$3l + l' - 3D$	·6	7	50	14·8
$3l - l' - 3D$	·7	8	35	4·7
$2F - 2D$	6	10	102*	17·38
$2F + l - 3D$	1	8	38*	9·2
$2F - l - D$	7	9	152	15

Double-Entry Solar Terms—continued.

Second Argument.	Max. Coef.	No. of Terms.	No. of Parts.	Monthly Addition.
$2F+2l-4D$	$4''$	8	94	29·5
$2F-2l$	2	7	63	1·7
$2F+l'-2D$	3	7	80	20·1
$2F-l'-2D$	2	5	48	4·3
$2F+l+l'-3D$	·3	4	26	8·4
$2F-l-l'-D$	·5	4	28	·5
$2F+l-l'-3D$	·4	3	31	5
$2F-l+l'-D$	·4	4	39	7
$2F+3l-5D$	·3	4	28	10·8

It is estimated that these 22 tables will require from 100 to 110 pages. The average number of terms per table is nine.

52. The next table shows the planetary terms which are put into double-entry tables.

The first argument is l': the second arguments are T − V for Venus, T − M for Mars, and J for Jupiter, where the symbols represent the mean longitudes of the four planets.

The terms are either independent of l, D, F, or are expressed as coefficients of the cosine and sine of l, 2D, $2D-l$.

The maximum coefficient in each case and the number of terms included are shown.

Synopsis of Planetary Terms.

Coef. of	Venus. Max. Coef.	Venus. No. of Terms.	Mars. Max. Coef.	Mars. No. of Terms.	Jupiter. Max. Coef.	Jupiter. No. of Terms.
1	$0''·9$	27	$0''·3$	22	$0''·7$	14
cos l	·3	33	·2	24	·3	19
sin l	·1		·01		·2	
cos 2D	·2	37	·05	16	·2	18
sin 2D	·1		·04		·2	
cos $(2D-l)$	·7	38	·07	25	1·2	22
sin $(2D-l)$	·7		·05		1·2	

These would form 21 double-entry tables, with an average of 13 terms per table. But it is a question whether some or all of the last 5 tables for Mars should not be included amongst the remainder terms. The 21 tables would require from 30 to 40 pages.

The Remainder Terms.

53. The terms which have not been included in any of the regular tables, and which are called above "remainder terms," are classified in the table which follows according to their approximate periods.

P_0 denotes the terms whose periods are greater than two months, and P_n those which have periods approximately equal to $1/n$ of a month.

The number of terms and the maximum coefficient in seconds of arc in each set are given, and the solar terms are separated from the planetary.

	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
Solar, No. of Terms .	6	12	11	11	8	4	2	1	0	2
,, Max. Coef. .	·08	·06	·04	·10	·06	·03	·01	·02	...	·01
Planetary, No. of Terms	14	23	28	27						
,, Max. Coef. .	·05	·06	·02	·03						
Total No. of Terms	20	35	39	38	8	4	2	1	0	2

The proposal in No. 24 was that the terms in P_0, P_1, P_2 should be computed for every ten days, by the device of separating the short-period terms as factors of the cosine and sine of l, $2D$. It is an open question whether the same device should not be applied to P_3, P_4, the rest of the terms being neglected; the error caused by so doing has a maximum value of $0''.1$.

Each of the ten-day tables would require 4 or 5 pages if extended to 200 years.

The semi-annual remainder terms (No. 29) are not included above; there are about 30 of them. One page will be sufficient to give them for ± 300 years from the epoch (1900·0).

Computation of the Longitude from the Tables.

54. In order to give some idea of the nature and amount of the work which will be required to obtain the ephemeris of the Moon in longitude, a skeleton form is made out below. No really systematic arrangement of the computing sheets has been attempted, and such would be premature at the present stage. A sufficiently close idea of the work will, however, be obtained without difficulty, especially by any one familiar with Hansen's tables and their use.

It will be seen that the computation of an ephemeris rather than that of a single place is the main object. For a single place, however, the work is not very different. The double-entry tables at ten-day intervals require the computation of two or three consecutive places to be made, while those at half-day intervals will need three or four places, according to the degree of accuracy required. In general, however, the errors of early observations demand a lower degree of accuracy for a single place than for the modern ephemeris. It is expected that the interests of the computer of a single place for dates in the past may be considered by certain omissions and adjustments of the added constants, for which instructions can be given.

In the preparation and use of the forms, labour can be saved in three ways. The forms can be printed so as to serve for the 730 half-days of the year, and on them certain quantities which are the

same at corresponding dates of every year can be printed. Numbers which remain the same for a given year or for a great many half-days can be inserted with a rubber stamp. In obtaining the arguments of the double-entry tables, the adding machine with the usual printing device and a special tabulating attachment (now being manufactured) will save much labour. Indeed, it is quite possible that the adding machine is the proper means of performing all the additions if economy and accuracy are to be fully considered. It has already shown its value in the formation of the arguments for the construction of the tables.

The whole work described in this paper can perhaps be summed up in a single statement, namely, that the average computation of a place in longitude for ephemeris work is equivalent to the addition of between 60 and 70 numbers per half-day when the mean arguments have been obtained. It may be mentioned that there are no multiplications or interpolations demanding more than three figures. On the whole, in spite of the insertion of so many more terms than are contained in Hansen's tables, I believe that the ephemeris work will demand little if any more time from the computer, and possibly less.

The Longitude.

55. The longitude requires the following computations:—

(*a*) Arguments for the beginning of the year, as in Hansen, but with D instead of l as the first argument of the double-entry tables.

(*b*) Arguments of double-entry tables for the beginning of each month (as in Hansen, but with exact numbers: use adding machine). Minute monthly addition to a few arguments for figure of Earth terms (No. 31).

(*c*) Interpolating factors of single-entry terms at beginning of year (constant through the year). Add monthly changes for figure of Earth terms (No. 31), except for arguments l, 2D, 2D $-$ l.

(*d*) *Arguments* l, 2D, 2D $-$ l: *interpolating factors; 10-day intervals.*

Long-period terms (semi-annual table) . . .	...
Large Venus term (single-entry with proper factor) .	...
Nineteen-year terms „ „ „ .	...
Double-entry tables for Venus, Mars, Jupiter .	3 Nos.
Remainder terms (for l, 2D)	...
Diff. from tabular argument (*c*): stamp . . .	...
Sec. change from beginning of year + adjustment of arg. (No. 41) − constants added to tables } (printed)	...
Sums: A for arg. l; B for arg. 2D; C for arg. 2D $-$ l	...

Interpolate to halves with 2nd differences, then to tenths.

(*e*) *Factors of tabular values, args. l, 2D, 2D − l; 10-day intervals.*

Double-entry tables for Venus, Mars, Jupiter	3 Nos.
Remainder terms (for l, 2D)	...
Secular change of coef. (for 2D, 2D − l): (stamp)	...
Subtract added constants (printed)	...
Sums: E for arg. l; F for arg. 2D; G for arg. 2D − l	...

Interpolate to halves with 2nd differences, then to tenths.

(*f*) *True longitude; 10-day intervals.*

Long-period terms (semi-annual table)	...
Large Venus term	...
Nineteen-year terms	...
Double-entry tables for Venus, Mars, Jupiter	3 Nos.
Remainder terms	...
Sum =	a_0

Interpolate to halves with second differences.

True longitude; half-day intervals.

(*g*) *Single-entry tables.*

Args. l, 2D, 2D − l (nearest tabular args.)	3 Nos.
Args. 2l, 2D, 4D − 2l, 2D + l, 4D − l, 2D ± 2l	7 Nos.
Sum =	a_1
Products of E, F, G (see (*e*)) by first three numbers in (*g*) (Crelle)	3 Nos.
Products of A, B, C (see (*d*)) by variations per $0^{d}{\cdot}001$ in tables for l, 2D, 2D − l (Crelle)	3 Nos.
Constant added to make positive (printed)	...
Sum =	a_2
Args. l', 2D ± l', 2D − l ± l', l ± l', 2D + l − l'	8 Nos.
Sum	...
Product of sum by k (t − 1900) (Crelle)	...
Sum =	a_3
Args. 2F, 2F − 2D, 2F ± l	4 Nos.
Sum	...
Product of sum by $2\delta\gamma/\gamma$ (single-entry table)	...
Sum =	a_4

(*h*) *Double-entry tables for tabular values of* D. *True longitude.*

Second arguments l', $l \pm l' - D$, $2l \pm l' - 2D$, $3l \pm l' - 3D$, $l \pm 2l' - D$, $2F \pm l' - 2D$, $2F + l \pm l' - 3D$, $2F - l \pm l' - D$	15 Nos.
Sum	...
Product by k $(t - 1900)$ (Crelle) . . .	...
Second arguments $l - D$, $2F + l - 3D$, $2F - 2l$, $2F + 3l - 5D$, $2F - 2D$, $2F - l - D$, $2F + 2l - 4D$	7 Nos.
Sum of last three, less (printed) const. multiplied by $2\delta\gamma/\gamma$	...
Sum of double entry values	...
Proportional parts to calendar dates (using 3rd differences)	...
Sum	...
a_0 interpolated from (f)	...
a_1, a_2, a_3, a_4	4 Nos.
Remainder terms (half-daily table) . . .	...
Sum = true less mean longitude (in seconds) . .	...
Conversion to degrees	...
Mean longitude at beginning of year (stamp) . .	...
Motion of mean longitude from beginning of year (printed)	...
Sum = true longitude (degrees)	...

Plans for Tabulating the Parallax.

56. The sine-parallax may be tabulated directly. I find that 5 single-entry and 14 double-entry tables will leave outstanding—

2 terms with coefficients 0″·0090, 0″·0066;
4 „ „ „ between 0″·0040 and 0″·0020;
10 „ „ „ „ 0″·0020 „ 0″·0005.

It seems worth while to add a few more double-entry tables which will include the majority of these terms, especially as all the arguments used in the parallax have already been found for the longitude. Five more double-entry tables will include all terms greater than 0″·0014, and leave outstanding known terms the sum of whose coefficients is less than 0″·0070—a quantity less than the final degree of accuracy, namely, 0″·01.

The tables should be computed and arranged like those for the longitude, except that the computations are carried to 0″·0001 and printed to 0″·001.

57. Hansen has 5 single-entry and 14 double-entry tables, and, in addition, the elliptic table with the variable argument z and the reduction from the logarithm of the sine-parallax to the parallax. He omits one term with a coefficient of $0''{\cdot}0070$, twelve terms with coefficients between $0''{\cdot}0040$ and $0''{\cdot}0020$, and very numerous smaller ones.

The work of the ephemeris computer would be slightly greater with the new plan, owing to the inclusion of the small terms.

The Tabulation of the Latitude.

58. A few further details beyond the suggestions mentioned in No. 9 may be given. If the plan there suggested be adopted, it will involve the construction of 6 single-entry and 4 or 5 double-entry tables with new arguments, and about 20 double-entry tables with arguments the same as those occurring in the longitude. These will involve three sets of interpolations from the values at times when $D = 0^d{\cdot}0, 0^d{\cdot}5, \ldots$ to the actual times required. There will further be a considerable number of remainder terms, which may have to be dealt with in a manner similar to the plans for the corresponding terms in longitude. There will be less computation than in Hansen owing to the smaller number of new arguments required. On the other hand, there are three sets of double-entry tables as against one in Hansen's plan, and many more terms must be included. An increase of 25 to 50 per cent. in the ephemeris computation is to be expected.

Corrections to the Constants.

59. In these tables for the Moon's place it should not be forgotten that the arbitraries of the solution may require changes, and that it should be possible to make such changes easily. For a change in the eccentricity, only the two largest single-entry tables in longitude will probably be affected, and these only by a constant factor which can in most cases be included by constant additions to A, B (No. 55). For a change in the inclination, the latitude will require multiplication by a constant factor, perhaps solely by a constant addition to the C of No. 9. A change in the ratio of the solar and lunar parallaxes will scarcely extend beyond the single-entry table with argument D. A change in the lunar parallax can be made by a constant factor for the true parallax. Any changes in the arguments will be made solely in the tables for the arguments.

Order of Computation of the Tables.

60. It is therefore possible to make final tables at the outset for all quantities except (1) the arguments of the larger terms, (2) the single-entry terms which depend on the solar parallax, (3) the

figure of the Earth terms (the observed constant requires extended investigation for any increase in the accuracy of its determination), (4) the single-entry table in parallax, combining the constant of parallax with the elliptic terms. It is hoped that after the preliminary tables have been made, an extended comparison with Hansen's places, and thence with observation, may be undertaken. After the constants have been thus corrected there will probably remain residual errors, which will be analysed, if possible, into periodic and secular terms which are to be incorporated into tables, and which will form definite empirical corrections awaiting a theoretical explanation. The rapid and accurate methods of Cowell, as shown in his series of papers on the lunar theory, are available for the analysis, whilst Newcomb's forthcoming work on the occultations since the beginning of the seventeenth century will form a valuable basis for the long-period residuals as soon as the corrections to Hansen's places have been made.

New Haven, Conn.:
1909 *November* 2.

Celestial Photographs taken at Mount Wilson Solar Observatory, California. By Professor G. E. Hale, D.Sc.

[*Extract from a letter to the President.*]

I am sending you a few photographs recently taken here, which may possibly be of interest to the members of the Astronomical Society.

1. Enlargement of a prominence, 85,000 miles high, made by Mr. Ellerman, August 21, 1909, $6^h\,22^m$ a.m. This was made with the Snow telescope and 5-foot spectroheliograph, using the H_α line. It is not especially remarkable, but interesting as one of our best prominence photographs.

2. Enlarged photographs of Mars,* made by myself with the 60-inch reflector on October 5 and November 3, 1909. The instrument was arranged in the Cassegrain form, with equivalent focal length of 100 feet. A pneumatic shutter from a photographic lens was mounted in a wooden support immediately in front of the plate-holder. Between exposures, the telescope was displaced in right ascension or declination by the slow-motion electric motor, so that a series of images was obtained on the plate. A Seed "Process" plate was employed, sensitised for the red by Wallace's "Pan-iso" formula. In front of the plate was a Wallace red screen, transmitting light of wave-length greater than λ 5800. Such a screen is necessary to increase the contrast. The exposure was about two

* These photographs are being issued in the Society's series of Reproductions of Celestial Photographs.

seconds. The glass positives sent to you are enlarged about six diameters from the original negatives.

3. As it seemed to be of interest to have some check on the quality of the Mars photographs, I made a few exposures on the Moon, with the same arrangement of the telescope, on the night of October 5. In this case a Seed No. 23 plate was employed without a red screen, and the exposure was about one second. In making the transparencies sent you, the degree of enlargement is the same as in the case of the Mars photographs.

I have devoted only a few evenings to this photographic work on Mars, and the results are not nearly equal to those that could be obtained with special apparatus. The aperture used for both the Moon and Mars was 44 inches, as it seemed best to cover the edge of the mirror, which is sometimes slightly turned up before midnight.

Visually, the performance of the 60-inch reflector, especially when used in the Cassegrain form, is all that could be desired. On November 3, when the seeing was very fine, a power of 800 was used on Mars to excellent advantage. With this power, small details, not visible with lower powers, were brought out clearly and distinctly. As for the appearance of the image, Barnard's description of his observations with the 36-inch Lick refractor, *Monthly Notices*, volume lvi. page 166, fits the case perfectly.

Pasadena, California:
1909 *November* 6.

Terrestrial Reproduction of the Spectra of the Tails of Recent Comets. By A. Fowler. (Plate 8.)

Introductory.

Spectroscopic observations of comets prior to those of Daniel's Comet (*d* 1907) appear to have indicated the prolongation into the tail of the bands of carbon which have long been recognised as the chief characteristic of the majority of comets. In the case of Daniel's Comet, however, Deslandres * announced that while the head showed the ordinary cometary spectrum, the tail exhibited a different spectrum, consisting of three lines or bands of unknown origin at approximate wave-lengths 402, 426, and 455. These new bands, as shown by photographs taken with a prismatic camera, were feeble in the head of the comet, but could be traced in the tail to a distance of three-quarters of a degree.

A similar result was obtained by Evershed † from photographs taken with a prismatic camera at Kodaikánal; the strong cyanogen band 3883 was almost restricted to the head, while three radiations,

* *Comptes Rendus*, cxlv. p. 445 (August 1907).
† *Monthly Notices R.A.S.*, vol. lxviii. p. 16 (November 1907).

at approximately 4015, 4260, and 4553, extended about one and a half degrees into the tail. Evershed also noted three other faint radiations about 358, 369, and 378, which extended nearly half a degree from the head.

The three principal new radiations, and some fainter ones, were also photographed in Daniel's Comet by Chrétien * at Nice, and the more powerful instrument employed made the additional revelation that each of the three new bands was double. The wave-lengths given by Chrétien were 4012, 4020; 4258, 4275; 4503, 4542.

The new bands were also a prominent feature of the spectrum of Morehouse's Comet (*c* 1908), of which extensive observations were made by several astronomers in Europe and America.†

Some of the photographic records of the spectrum were made with slit spectrographs, and others with prismatic cameras, the latter having the advantage of showing the constitution of the tail as well as that of the head in a single exposure. In nearly every case the dispersion was small, and the accounts of the various photographs show discrepancies in regard to details, into which it is scarcely necessary to enter.

The facts which stand out prominently are—(1) That the new radiations first observed in Daniel's Comet were among the principal features of the spectrum of Morehouse's Comet, and could be traced as far as eight degrees from the head (Deslandres); (2) That the three bands in question were clearly double, as found by Chrétien in Daniel's Comet; (3) That the separation of these and of other fainter doubles was approximately proportional to the wave-length, and remained about the same during the period of observation; (4) That a line or band at 3914, which Deslandres identified with the chief negative pole band of nitrogen, was also prominent, and showed no duplication; (5) That the cyanogen band 3883 was well developed in the spectrum of the nucleus, but did not extend far into the tail; (6) That the "carbon" bands 4736, 5165, and 5636 were also probably present in the spectrum of the head of the comet.

Terrestrial Reproduction of the Spectrum.

A systematic search for a terrestrial source of the new cometary bands formed part of my programme of work for the current session, and some interesting results have already been obtained. The investigation is still far from complete, but a short account of the results will be of value in indicating the true character of the cometary spectrum, as to which there has been a good deal of misconception.

* *Comptes Rendus*, cxlv. p. 549 (September 1907).

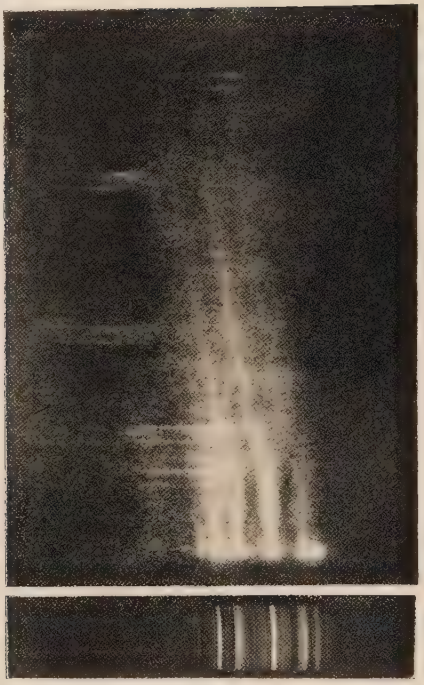
† Pluvinel, *Comptes Rendus*, cxlvii. p. 666 and cxlviii. p. 759. Deslandres, *Comptes Rendus*, cxlvii. p. 774 and p. 951; cxlviii. p. 805. Campbell and Albrecht, *Astrophys. Jour.*, vol. xxix. p. 84. Curtis, *Lick Observatory Bulletin*, No. 163. Frost and Parkhurst, *Astrophys. Jour.*, vol. xxix. p. 55. Rosenberg, *Astrophys. Jour.*, vol. xxx. p. 267.

In view of the probable low density of gases in the tails of comets, the most promising field of investigation seemed to be that of the kathode spectra of various compounds of carbon and other gases at very low pressures.

The first procedure was to re-examine a series of small photographs taken by Mr. Howard Payn and myself some years ago, in which the kathode glow was specially observed in tubes initially filled with hydrogen, prepared by heating potassium formate and potash. Among these photographs was one taken on March 12, 1902, which differed greatly from all the others, and formed the subject of several experiments at the time, without any definite result as to the substance or substances to which the bands were to be attributed. It is now evident that this photograph is almost an exact replica of the spectrum of the tail of Morehouse's Comet in the region over which it extends, namely, 3800–4860. Not only do the various bands agree in position with those of the comet, but the relative intensities are identical throughout, within the limits of the error of estimation, if the tail alone be considered. This identity as regards the principal bands—omitting that of cyanogen, which is but feebly represented in the terrestrial spectrum—is shown in the appended table. The measurements of the comet spectrum are those which seem to be entitled to the greatest weight from a consideration of the instruments employed. In the case of the terrestrial spectrum, which is only 1·48 cm. in length from 3883–4861, the wave-lengths were deduced by a Cornu-Hartmann formula depending on the first head of the cyanogen band 3883, and faint lines corresponding to H_β and H_γ.

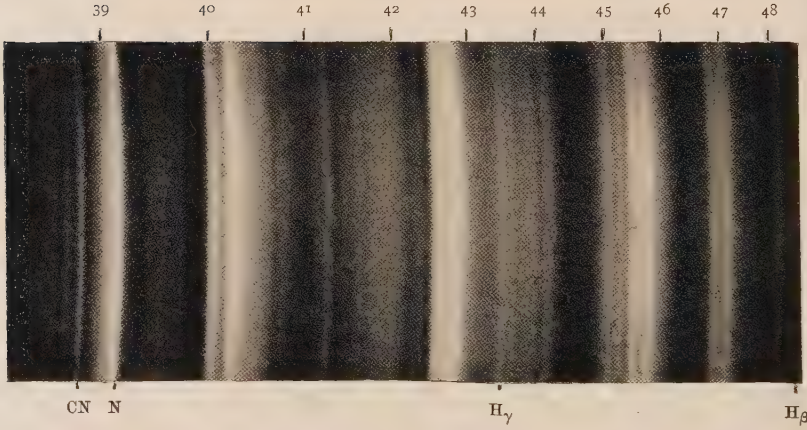
The band at 3914, under the conditions of experiment, appears like a rather broad line, practically sharp at both edges, while the other bands are either like broad lines with ill-defined edges, or have the appearance of short flutings fading towards the red. The middle parts of the apparent broad lines were measured, and the wave-lengths may be a tenth-metre or more in error. In each case the figures in brackets are estimated intensities, and it is particularly to be noted that the wave-lengths in the terrestrial spectrum are those of *all* the principal bands shown on the photograph.

Morehouse's Comet.				Terrestrial Spectrum.
Deslandres.	Campbell and Albrecht.	Curtis.	Mean.	
4714·0 (1)	4714·9 (5)	4716·3	4715·1	4714·1 (3)
4688·5 (4)	4690·2 (7)	4690·7	4689·8	4688·5 (3)
4570·5 (5)	4570·2 (3)	4570·2	4570·3	4569·9 (5)
4548·1 (6)	4549·4 (4)	4545·9	4547·8	4545·6 (6)
4275·8 (10)	4275·6 (10)	4276·0	4275·8	4276·5 (10)
4255·2 (9)	4254·9 (8)	4254·0	4254·7	4252·6 (8)
4021·8 (9)	4022·2 (8)	4021·3	4021·8	4021·0 (8)
4003·3 (8)	4002·5 (6)	4002·1	4002·6	4001·3 (6)
3913·2 (7)	3914 (2)	3914·1	3913·8	3914·8 (9)



(a)

(b)



(c

(a) MOREHOUSE'S COMET, photographed with a prismatic camera by Frost and Parkhurst, 1908 Oct. 29.

(b) Kathode spectrum of probable carbon compound at very low pressure (Fowler).

(c) Enlarged copy of (b), with scale of wave-lengths.

The agreement in wave-lengths is even better than could have been expected, considering the small dispersion which was of necessity employed for the comet spectrum. The general accordance is probably in part due to the fact that the dispersion was of the same order in the two cases, because the appearances of bands and flutings vary greatly with the scale of the photographs, as do also the positions of the parts measured as heads.

As the divergences in the intensities of the least refrangible pair are probably sufficiently accounted for by partial superposition of the blue carbon band 4736 in the head of the comet, the evidence must be considered conclusive as to the identity of the terrestrial spectrum with the new cometary bands.*

The original photograph showing these bands is perhaps of sufficient interest for reproduction. It is shown on Plate 8, c, enlarged six times from the negative.

A comparison with a photograph of the cometary spectrum is given in the same plate, a and b. The comet spectrum has been copied from a reproduction of Frost and Parkhurst's photograph of Morehouse's Comet taken with a prismatic camera on October 29, 1908,† and has been slightly enlarged so as to approximate closely to the scale of the terrestrial spectrum, which is reproduced on the scale of the original. The duplication of the bands is not shown in the spectrum of the comet on account of the overlapping of the images, but the general correspondence of the two spectra is evident at a glance. The first knot on the right in the spectrum of the head of the comet appears to include the carbon band 4736, and is consequently brighter than the corresponding double in the terrestrial spectrum; in the tail, however, it has an appropriately lower intensity. The fifth knot, as remarked by Frost and Parkhurst, includes the band 3914 and the cyanogen band 3883, but only the former extends into the tail. Hence, so far as it can be shown from photographs of this kind and on this scale, there is an exact correspondence between the terrestrial spectrum and the spectrum of the comet's tail.

The spectrum in question was obtained from a tube about an inch in diameter, having a sparking distance of about 8 inches, filled at first with what was believed to be pure hydrogen, and then exhausted to a pressure low enough to produce fluorescence of the glass. The kathode in this tube was made in the form of a nearly complete ring of platinum wire, as described by Sir William Crookes,‡ such that it could be heated by a current from a battery, with a view to driving out the occluded gases. The spectrum was photographed with the tube end on, but the laboratory note-book does not state whether the kathode was separately heated when the photograph in question was taken. It is recorded by Mr. Payn,

* Another double band at 3799, 3783, seen faintly in the original photograph, appears more strongly in later photos taken with an instrument better adapted for this part of the spectrum. This also appeared in the comet.

† *Astrophysical Journal*, vol. xxix., plate v.

‡ *Roy. Soc. Proc.*, vol. lxix. p. 402 (1902).

however, that "the tube was at the muddy stage," and, in further explanation, that "'muddy' means continuous glow after all pink buttons and pink glow have passed away; it shows that very little hydrogen is in the tube, and all sorts of other things may be present; plate very curious."

Experience in work at low pressures, in common with that of others, had in fact shown that the final spectrum obtained did not depend so much upon the gas introduced as upon minute quantities of nitrogen and carbon compounds, which are exceedingly difficult to eliminate. Hence it was not considered that hydrogen was necessarily responsible for the spectrum in question.

Attempts to trace the origin of the Spectrum.

On resuming the investigation, the kathode glow was first observed and photographed in several ordinary sealed vacuum tubes, with the only result of showing that the band at 3914 in the cometary spectrum was in all probability properly assigned to nitrogen by Deslandres. It also seemed possible that a second kathode band of nitrogen of rather lower intensity, at 4279, might account for the cometary band 4276, but later experiments showed that the origin of the latter could not be wholly nitrogen. These two bands are almost the sole representatives of nitrogen when the pressure is very low.

The original experiment was next repeated, except that the platinum wire ring was replaced by an aluminium electrode in the form of a washer, the outside diameter of which was nearly the same as the internal diameter of the tube, and the central opening about half an inch diameter, so as to give a clear end-on view of the illuminated gas. Hydrogen was prepared in the same manner as before. When the pressure was about 0·2 mm. the two spectra of hydrogen were observed, but the secondary spectrum completely disappeared, and the line spectrum was restricted to the negative electrode, when the pressure was reduced to about 0·01 mm.; at this stage the new bands came into view as in 1902. Other bands, commonly designated the "carbonic oxide" bands, were also present, but the result was satisfactory as showing that the spectrum could be reproduced when desired

The presence of the carbonic oxide bands suggested that the new spectrum might depend upon the presence of oxycarbons at very low pressures. Dried carbon dioxide was accordingly introduced into the discharge-tube. At pressures ranging from about 0·2 to 0·04 mm., when the discharge was brightly luminous, the spectrum showed the flutings of carbonic oxide at 561, 519, 483, 451, etc., but when the pressure was reduced to 0·01 mm. the discharge was very feebly luminous, and the familiar carbonic oxide flutings were almost wholly replaced by the new bands. The chief result of this experiment was to show that the new spectrum could be obtained at least as easily after the admission of carbon dioxide

as after filling the tube with hydrogen, and that the intensity of the nitrogen band 3914 could be greatly reduced by washing out the tube with the carbon dioxide. The additional important fact was noted that 4276 did not tend to vanish with 3914, and must therefore be regarded as part of the new spectrum; in the spectra of comets, however, it may be combined with the kathode band of nitrogen at 4279.

Before proceeding further, it was considered necessary to make a control experiment in which the "empty" discharge-tube was directly connected to the pump. At comparatively high pressures the spectra of nitrogen and hydrogen were alone observed, but, after a little sparking and continued exhaustion, the carbonic oxide bands made their appearance, and finally, when the pressure was reduced to about 0·01 mm., the new bands were a prominent feature of the spectrum. The passage of the spark doubtless liberated gases from the glass and electrodes, or there might have been a slight leakage of air containing carbon dioxide. Hence, although the tube with an aluminium kathode might be considered a convenient way of obtaining the new spectrum, it was not suitable for investigations relating to the precise origin of the spectrum.

Further experiments were accordingly made with other discharge-tubes, one having the kathode made of magnesium, and two of platinum, of the same form as before. Aluminium and magnesium have an advantage over platinum, inasmuch as they do not blacken the tube, and magnesium was selected as a possible means of eliminating residual nitrogen by the union of the two elements.

In each case the empty tube was first tested; and although the nitrogen spectrum was predominant at comparatively high pressures, the new spectrum was observed when the spark had passed for some time and the pressure was sufficiently reduced. Washing through with air freed from moisture and carbon dioxide, together with sparking, quickly removed the gas from the tubes with platinum kathodes, but the tube with magnesium, though freed for a time, was less trustworthy in this respect. The most significant results were accordingly those obtained with platinum electrodes.

In some of the experiments, beginning with air, only the spectra of nitrogen and hydrogen were observed, even at the lowest pressures giving reasonable luminosity, and hence it may be considered that the new spectrum is not due to either of these elements. It was similarly found that oxygen did not yield the new bands at any pressure.

The fact that the new spectrum was never observed at the lowest pressures unless the carbonic oxide bands had been previously observed at higher pressures indicated that the most probable source of the new spectrum was a compound of carbon. Compounds of carbon with hydrogen, nitrogen, and oxygen, in the form of marsh gas, cyanogen, carbon monoxide, and carbon dioxide,

were therefore investigated. When the discharge was brightly luminous, the carbonic oxide flutings were visible in each case, and the new bands appeared when the pressure was so reduced that only slight luminosity remained (about 0·01 mm.). They were, however, observed with least contamination when the gas was carbon monoxide, and it should be remarked that the spectrum was not like that which appeared in the region of the kathode at somewhat higher pressures.

It cannot be asserted that all traces of oxygen were absolutely removed in the experiments on marsh gas and cyanogen, but, so far as they go, the observations agree with those of Lockyer * in suggesting that the so-called carbonic oxide flutings may be one of the spectra of carbon itself. In that case the new cometary bands should also be assigned to carbon, and the exact compound of carbon from which the spectrum is derived in comets may be indeterminate. The presence of nitrogen in Morehouse's Comet would suggest that the new bands might be derived from cyanogen if it were not for the fact that the nitrogen band was not recorded as accompanying them in Daniel's Comet.

The outcome of the experiments so far carried out is to indicate that the spectra of the tails of Daniel's and Morehouse's Comets were chiefly derived from some compound of carbon, under conditions which have only been reproduced in the terrestrial gases when at pressures of about one hundredth of a millimetre or less. The difficulties experienced in attempting to determine the precise origin of the spectra given by carbon compounds, even in more familiar circumstances, are so great that the Swan, or candle-flame, spectrum is variously attributed to carbon, hydrocarbon, and carbon monoxide by different authorities. Further work is therefore necessary; but it may be remarked that the circumstances under which the new carbon spectrum has been produced are in excellent agreement with the generally accepted views as to the physical conditions which prevail in the tails of comets.

The author has pleasure in acknowledging the valuable assistance given by Mr. Howard Payn in connection with the experiments made in 1902, and by Mr. H. Shaw in the course of the more recent work.

* *The Meteoritic Hypothesis*, p. 44.

Note on Variations in Brightness of Halley's Comet.

By Rev. T. E. R. Phillips, M.A.

Since 1909 November 16 I have observed Halley's Comet with a $12\frac{1}{4}$-inch Calver reflector on every available occasion, and the observations show some considerable variations in the comet's brightness. The aspect of so delicate an object is of course specially liable to be affected by atmospheric conditions, but the observations which have been made on ten nights seem sufficiently numerous to practically eliminate error arising from this source. I have commonly adopted the method of limiting apertures in comparing results, and have allowed for the atmospheric conditions prevailing at the time.

The comet appeared unquestionably brightest on the night of November 22, when it could be seen with a stop $3\frac{1}{2}$ inches in diameter placed eccentrically so as to avoid the obstruction of the flat. The Moon was nearly ten days old at the time and the atmosphere rather hazy, but the comet was surprisingly brighter than on the previous night, when the conditions were more favourable. Mr. Hollis, on the same night (November 22), using the 10-inch guiding telescope of the Astrographic Equatorial at Greenwich, estimated the brightness as about magnitude 10. The following night the comet was very inconspicuous with the full aperture of the reflector, and the conditions were not so inferior to those of the 22nd as to be held accountable for the whole change. The comet was not seen again till the end of the month, but since then it has been repeatedly observed under quite satisfactory conditions without regaining the brightness of November 22. On December 3 and 8 it could just be glimpsed on a dark sky with $3\frac{1}{2}$ inches of aperture, but on December 6, when the sky was very clear and transparent after rain and the stars brilliant, it was quite invisible with a $4\frac{1}{2}$-inch eccentric stop. It was then probably as faint as 11·5 or 12 magnitude. On December 8 the comet was brighter again, and was estimated to be about 10·5 or 11 magnitude.

The fact that the comet was most easily seen under the inferior conditions of November 22, and especially difficult on December 6 when circumstances were apparently excellent, may be regarded as proving the reality of changes in its visual brightness.

The distance between the Earth and the comet has been steadily diminishing, though less rapidly than formerly, so that the unexpected brightness on November 22 was probably in the nature of a spurt. It has, however, been generally brighter than the magnitude based on its early photographic images, so its recent comparative faintness need not be held to indicate a feeble display near perihelion. The spurt on November 22 suggests that the reverse is possible, if not probable.

Ashtead:
1909 *December* 9.

Occultation of Saturn by the Moon, 1910 March 13, visible in parts of South Africa. By A. M. W. Downing, D.Sc., F.R.S.

At JOHANNESBURG the emersion of Saturn is visible

	d	h	m
Local mean time 1910 March . .	13	6	39·0
Standard time ,, ,, . .	13	6	46·7
Angle from north point of Moon's limb .			260°
Angle from vertex of Moon's limb .			138

These times are for the centre of Saturn. The emersion of the outer Ring occurs $1^m \cdot 3$ earlier.

The immersion takes place 41^m before Sunset.

At DURBAN the emersion of Saturn is visible

	d	h	m
Local mean time 1910 March . .	13	6	47·1
Standard time ,, ,, . .	13	6	43·1
Angle from north point of Moon's limb .			252°
Angle from vertex of Moon's limb .			130

These times are for the centre of Saturn. The emersion of the outer Ring occurs $1^m \cdot 2$ earlier.

The immersion takes place 36^m before Sunset.

1909 *December* 7.

Burnham's Nebula near 63 Cygni (N.G.C. 7026). By the Rev. T. E. Espin, M.A.

This nebula was detected by Burnham with a 6-inch refractor and suspected to be double. Subsequently it was observed by him with the 18½-inch at the Dearborn Observatory, and in *Mem. R.A.S.*, vol. xliv. p. 216, measures are given as follows:—

Star and neb.	P 243°·1	D 26″·63	1878·468
	245·0	28·05	·476
	88·5	2·57	·476 distance of nuclei.

And on p. 295 he mentions "an excessively faint star, which with the nebula and the 9 mag. star makes a nearly equilateral triangle." In *M.N.*, vol. lii. p. 45, measures are given with the 36-inch of the Lick Observatory of the distance between the nuclei as follows:—

P 272°·9 D 6″·45 3 nt. 1891·57

In the note following he says that one of the nuclei is brighter than the other, and that they are not stars, but small enough for fairly accurate bisection; and further adds "that some rough measures

were made a few years ago which differ much from the results given here, but it is not probable that any change has occurred in the nebula." As far as I am aware, there are no other observations of this object.

This autumn I have observed the nebula on several occasions. It consists of two elongated masses running N. and S., each of which has a nucleus. To my eye the preceding nucleus is slightly the brighter, and I have estimated them as 11·2, 11·4 mag. respectively. On November 15 it seemed to me that the south part of the nebula ends more abruptly than the north. On the same night a faint star was suspected at the south end, and attempts were made to measure it on November 19 and December 4, taking its position from the 9 mag. star. On all occasions the space between the two masses has seemed filled with nebulosity, and the object has appeared as one nebula with two nuclei. In Burnham's earlier measure the distance between them has probably been accidentally halved. I have made the following rough measures of the extension of the nebulosity.

1909 Nov. 10	Major Axis 15″·65	Minor Axis 10″·85
,, 13	17·25	10·40
,, 15	17·50	

In the spectroscope the object was observed on two nights, and shows the spectrum of a gaseous nebula.

The following are the measures I have made :—

	B C (nuclei).		
1909·857	P 96°·8	D 5″·70	
·865	94·7	5·95	
·870	93·8	5·65	
·881	93·2	5·55	
1909·868	94·6	5·71	4 nts.
	A B.		
1909·854	P 245°·1	D 30″·51	A 9·2 mag.
·857	248·2	29·50	
·865	245·5	30·46	
1909·859	246·3	30·16	3 nts.
	A C.		
1909·854	P 239°·2	D 24″·39	
·857	238·7	24·74	
·865	239·4	23·64	
1909·859	239·1	24·26	3 nts.

A a.

1909·865	P 312°·9	D 20″·64	Mag. 14·0	
·871	310·1	20·54	13·5	
·882	314·9	21·75	13·5	
·909	311·4	21·46	13·5	
1909·882	312·3	21·10	13·6	4 nts.

A b.

1909·8816	P 228°·8	35″·01	Mag. 14·0	
·9227	225·0	34·41	14·0	
·902	226·9	34·71	14·0	2 nts.

The only other stars I was able to certainly see consist of a pair of faint stars which I have measured as follows :—

D E.

1909·882	P 197°·7	D ″·..	Mags. 12·0	12·2	
·909	194·5	8·30			
·923	194·1	8·90	12·5	12·8	
1909·905	195·4	8·60	12·2	12·5	3 nts.

A D.

1909·882	P 299°·2	D 64″·73	
·909	298·5	64·56	
·923	297·3	...	
1909·905	298·3	64·64	3 nts.

The star A is B.D. + 47° 3289

h m s
R.A. 21 1 22·4 Decl. + 47° 16′·5 (1855·0).

Tow Law, Co. Durham:
1909 *December* 10.

Star Colours and Spectral Types.

By W. S. Franks.

I have recently been making an analysis of the Potsdam star colours, as given in the *Revised Harvard Photometry* (*H.A.*, vol. l.), to see how they compared with my own (*M.N.*, vol. lxviii. p. 672), and I think the results may possess some general interest. The colours of all stars in the northern hemisphere, down to 6·5 magnitude, are recorded, with the exception of a few variables. The groupings adopted in this investigation are shown in Table I.

TABLE I.

Contingency Table for Northern Stars (Potsdam Colours).

Spectrum Group.	a. White.	b. Yellowish-White.	c. Pale Yellow.	d. Yellow.	e. Orange.	f. Orange-Red.	Total.
Oe5 Intermediate type.	3	4	1	0	0	0	8
B to B5 Orion type.	73	111	9	0	0	0	193
B8 to A4 Sirius type.	470	1066	79	11	1	0	1627
A5 to F2 *α* Carinæ type.	26	351	140	12	0	0	529
F5 Procyon type.	2	64	42	6	1	0	115
F7 to G8 Capella type.	8	94	294	137	8	0	541
K Arcturus type.	1	36	381	490	66	1	975
K2 and K5 Aldebaran type.	0	2	14	88	16	0	120
Ma to Md Betelgeuse type.	0	0	12	129	38	0	179
N 19 Piscium type.	0	0	0	0	2	6	8
Total	583	1728	972	873	132	7	4295

The value of the coefficient of mean square contingency, which measures the average deviation of the two characters from independence (being zero if independent, and approaching unity [1·0] if causal), is $C_2 = \cdot 778$, as determined by Miss Julia Bell, M.A., whose valuable assistance I gladly acknowledge. The corresponding value for my table of 3497 stars was shown by Miss Bell to be $C_2 = \cdot 756$ (*M.N.*, vol. lxix. p. 108). The difference is not much beyond the limit of the probable error of a random sample, and these entirely independent results may be considered to corroborate each other. It ought, however, to be pointed out that the two lists of stars are not quite identical in their composition, because the Potsdam limit is 0° to +90°, whilst mine is −25° to +90°. Moreover, there are 1654 more stars, 6·0 to 6·5 magnitude, in the

Potsdam list than in mine over the same region; and, as Professor Pearson has suggested to me, it is highly probable that the colour spectrum relation is different for each magnitude group. And the distribution of the spectral types is not by any means uniform: for, although there are 800 more stars in the Potsdam total than in mine (4295 against 3497), there are *fewer* specimens of the O and N types and Helium stars (B group) than in my list, owing to many of these objects being south of the equator.

Having got so far, it occurred to me that a fairer comparison would be to take the stars common to each list—2641 in number—and subject them to the same analysis. But, whilst going through the Potsdam results, it had become evident to me that there was a certain peculiarity in them that required elucidation; which was the abnormal number of yellowish-white (b) stars, with a corresponding deficiency in white (a). Though this does not affect the colour-spectrum relationship as a whole, it does very materially affect the *position* in the scale. If any stars are *white* it is the Helium ones (B group); yet among these the Potsdam observers have 50 per cent. more *yellowish-white*! The Sirian stars, a large proportion of which are white to normal observers, come out still worse—there being above double the proportion of yellowish-white to white. This is altogether at variance with anything within my knowledge, and must, I think, be due to some peculiarity in the instrument or observer. The kind of instrument used, whether refractor or reflector, and also the aperture of the telescope, may exercise a perceptible effect on the ultimate apparent tint of a star. Thus I well remember that many stars looked distinctly paler in colour with my $11\frac{1}{4}$-inch reflector than in the 5-inch refractor. But the *eye* of the observer is, after all, the main factor, and it has often been a matter of remark, in my thirty years' experience of star-colour observation, how many otherwise good observers fail utterly in the correct estimation of star tints. Of course, like anything else, it is a gift that requires cultivation and practice; but, considering the close connection between colour and spectrum, and the great discrepancies that exist between photographic and visual magnitudes—almost entirely due to colour differences—it is surely a matter of importance that we *should* know the colours of stars with some degree of certainty? No one who has had occasion to critically examine some of the beautiful photographic star charts now available can fail to be struck with the fact that they do not represent the stars as they appear to the eye or in the telescope, and thereby they lose somewhat of their value. The discrepancies between actinic and visual magnitudes are both numerous and striking; it is therefore desirable to obtain all the information we can about the reconciling factor of colour. The combination of ordinary and isochromatic plates has been freely advocated as giving a complete answer to this problem; but, in the first place, the plates themselves vary much and cannot be implicitly trusted; and, secondly, stars of the same tint do not always give identical spectra. whilst others of similar spectra often show a

perceptible difference in tint to a practised eye. For instance, a white star may be of either Helium or Sirius type, and it is difficult to see how the photographic method of magnitude difference would discriminate between them. A yellowish-white star may belong to either the Sirius, *a* Carinæ, or Procyon type; pale yellow to the Capella or Arcturus type; yellow may range from the Capella up to the Betelgeuse group—and so on. Numerous examples of such are to be found both in the Potsdam and my own observations. Again, a star of F5 spectrum may be either yellowish-white or pale yellow; A—white or yellowish-white; K—pale yellow or yellow, etc. These variations from the normal type are real and well corroborated, and cannot be ignored or easily explained away. Having called attention to the Potsdam peculiarity *re* yellowish-white, it only remains to add that elsewhere there is a pretty close agreement between us. As a fair sample, I quote a few of the brighter stars to show what degree of accordance can be expected:—

Star.	Colour.		Spectrum.
	P.	Fr.	
α Andromedæ	white	white	A
α Ursæ Minoris	yellowish-white	yellowish-white *	F8
α Arietis	pale yellow	yellow	K2
α Persei	yellowish-white	yellowish-white	F5
α Tauri	yellow	orange-yellow	K5
α Aurigæ	pale yellow	pale yellow	G
γ Orionis	yellowish-white	white	B2
β Tauri	white	white	B8
β Aurigæ	,,	,,	A Pec.
γ Geminorum	yellowish-white	,,	A
α Geminorum	,,	greenish-white / white	A
α Canis Minoris	,,	yellowish-white	F5
β Geminorum	pale yellow	pale yellow	K
α Leonis	yellowish-white	white	B8
α Ursæ Majoris	pale yellow	yellow	K
η Ursæ Majoris	white	white	B3
α Boötis	pale yellow	orange-yellow	K
α Ophiuchi	yellowish-white	white	A5
α Lyræ	white	bluish-white	A
α Aquilæ	yellowish-white	yellowish-white	A5
α Cygni	white	white	A2

Perhaps the most noticeable case of difference in the foregoing list is that of Arcturus; most observers would, I imagine, rate this splendid star as something deeper than "pale yellow."

I may now utter a word of caution against accepting as definite the records of star tints from solitary observations; such are not

* This should really be Y's—that is, the intermediate grade between yellowish-white and pale yellow—on my scale; being the mean of 154 observations, by 21 observers (*Memoirs of B.A.A.*, vol. i. p. 43, and vol. v. p. 458).

to be trusted, unless corroborated by spectrum results—and many cases of apparent discrepancy may be set down to this cause. I am quite aware that this affects my own work as well as that of others; but I am convinced that only in the multiplicity of observations is accuracy likely to be attained. Therefore I had carefully entered the observations from my four catalogues (the last being included in the reports of the Star Colour Section of the *B.A.A. Memoirs*, vol. i. p. 33, and vol. ii. p. 109) in the margin of the *Revised Harvard Photometry*; these were made in different years—between 1877 and 1902—with different instruments, both refractors and reflectors, and with apertures ranging from 3 to $11\frac{1}{4}$ inches: we may thus assume that the instrumental effect was practically neutralised. Then I took the *mean* of all these observations combined with the Potsdam colours, and formed another contingency table (see Table II.) of 2641 stars, which were common to both lists; the final results being, I think, as reliable as can be expected by the method of eye estimation.

TABLE II.

Contingency Table for Northern Stars.

Mean Colours—Potsdam and Franks.

Spectrum Group.	White.	Yellowish-White.	Pale Yellow.	Yellow.	Pale Orange.	Orange.	Orange-Red.	Total
Oe5 Intermediate type.	4	4	0	0	0	0	0	8
B to B5 Orion type.	129	32	2	0	0	0	0	163
B8 to A4 Sirius type.	524	373	30	2	2	0	0	931
A5 to F2 *α* Carinæ type.	28	217	80	1	0	0	0	326
F5 Procyon type.	1	38	35	1	1	0	0	76
F7 to G8 Capella type.	0	41	221	45	11	0	0	318
K Arcturus type.	0	6	170	275	136	26	0	613
$\frac{9}{10}$K2 and K5 Aldebaran type.	0	0	3	31	33	20	0	87
Ma to Md Betelgeuse type.	0	0	2	32	34	45	1	114
N 19 Piscium type.	0	0	0	0	0	0	5	5
Total	686	711	543	387	217	91	6	2641

My own colour perception is, I believe, fairly normal; what slight bias I have is towards the red end of the spectrum, consequently it is extremely improbable that I should see star tints *whiter* than they really are; such would be a manifest contradiction. The colours given in the following table are practically synonymous with the Potsdam scale, though I prefer to subdivide

e into orange and pale orange, for it is surely as easy to decide whether a star is orange or pale orange as whether it is yellow or pale yellow? Of course these divisions are broad merely, and do not by any means exhaust the tints that are known among star colours. Such delicate shades as bluish- or greenish-white would be classed as white, creamy- or ruddy-white with yellowish-white, orange-yellow with yellow, etc.

I am inclined to think that there are more stars of a decidedly *bluish* tinge among the lower magnitudes, from the intense actinic effect which some small stars produce on photographic charts.

The value of the coefficient of mean square contingency for the above is $C_2 = \cdot 8360$, as deduced by Miss Bell.

This marks a distinct advance in the degree of relationship between star colours and spectral types, and is a proof in itself that the method of taking *mean* colour values is superior to any individual estimates alone. Whilst preparing the foregoing analysis, some obvious cases of discordance came to light, where the colours did not fit the spectra; these are set forth below.

Colour and Spectrum Discordances.

H.R.	Spectrum.	Potsdam Symbol.	Colour.	Observers.	Colour required by Spectrum.
589	B5	c	Pale yellow	P. and Fr.	a. White.
7763	B4	c	do.	do.	do.
7391*	A	d	Yellow / Pale orange	P. / Fr.	a. White, or b. Yellowish-white.
7759	A	e	Orange	P. and F_r.	a. White, or b. Yellowish-white.
8306	A	d	Yellow / Pale orange	P. / Fr.	a. White, or b. Yellowish-white.
8857	A	d	Yellow	P. and Fr.	a. White, or b. Yellowish-white.
2394	F	d	do.	do.	b. Yellowish-white.
2241	F5	a	White	do.	b. Yellowish-white, or c. Pale yellow.
3779	F5	e	Orange	do.	b. Yellowish-white, or c. Pale yellow.
7904	F5	d	Yellow / Pale orange	P. / Fr.	b. Yellowish-white, or c. Pale yellow.
996*	K	b	Yellowish-white	P. and Fr.	d. Yellow.
5853*	K	b	do.	do.	d. Yellow.
6638	K	b	do.	do.	d. Yellow.
6849	K	b	do.	do.	d. Yellow.
8729	K	b	do.	do.	d. Yellow.
5940	K5	c	Pale yellow	P. and Fr.	d. Yellow, or e. Orange.
4504	Ma	c	do.	do.	e. Orange.
4964	Ma	c	do.	do.	e. Orange.

* *Notes:*—

996—One of the discrepancies noted in Table VI., *H.R.* (p. 234)—Comparison between H and P—Visual spectrum A, which agrees with the colour.

5853—The same remark applies to this—Visual spectrum A, which agrees with the colour.

7391—In this case the visual spectrum is K, agreeing exactly with colour.

The above concrete examples account for most of the outliers in Table II.; it seems pretty certain that the photographic spectra are in fault.

Of course the comparison between star colours and spectra depends on *photographic* spectra only, which in some respects may not correspond exactly to the visual spectra; as Pickering says: "the difference may be due to the different portions of the spectrum observed visually and photographically." A duplication of the spectra on "isochromatic" plates would, no doubt, yield some interesting results, and might possibly modify the existing classification. In estimating star colours, too, we must also take into account individual errors of observation, which may often amount to one grade; thus a Y^1 (yellowish-white) might be rated at different times as O (white) or Y^2 (pale yellow)—but by taking the mean we arrive at its most probable value. It is much easier to determine the tints of some stars than those of others; this has also been remarked by several veteran observers. I have noticed that while many stars give identical results in my four catalogues, there are others in which no two estimates exactly agree — this emphasises the necessity of only adopting *mean* values.

Now if we eliminate these 18 discordant stars from Table II., we find the coefficient of mean square contingency is thereby increased to $C_2 = \cdot 8381$ (for 2623 stars). But, as the composition of the table so closely resembles that of Table II., it was thought unnecessary to give it *in extenso*. Judged by the spectrum criterion, we may now consider the colours of nearly all stars in the Northern hemisphere, down to 6·5 magnitude, to be fairly well determined. Would that we could say the same of the Southern hemisphere! But here, unfortunately, there is a great void—waiting for some southern Potsdam observer to fill.

As a rough guide to future work, I may just enumerate the very meagre results which we possess at present of star-colour work in the Southern hemisphere.

Observations of Southern Star Colours.

Dunlop, Paramatta, 1832? . . —Colours of about 530 stars —*Monthly Notices*, vol. ii. p. 190 (see note).

Tupman, Montevideo, 1864 . . —Colours of about 114 „ —*Monthly Notices*, vol. xxxiii. p. 312.

Franks, Leicester, etc., 1877–92 . —„ „ 900 „ between 0° and −25°—in four catalogues, three of which are in the Society's library, and the fourth published in the sectional reports of the B.A.A.— *B.A.A. Memoirs*, vol. i. part iii., vol. ii. part iv.

Williams, *Voy. to Australia*, etc., 1885–6—Colours of about 153 „ —"Cat. of Mags. of 1081 stars."

Gage, India, 1886 . . . . — „ „ 48 „ —*Liverpool Astron. Soc. Journal*, vol. v. pp. 138, 207.

Baracchi, Melbourne, 1889 . . —Colours of about 61 „ —*Monthly Notices*, vol. l. p. 66.

Eddie, Grahamstown, 1894 . . —Colours of about 100 stars.
—*B.A.A. Memoirs*, vol. v. p. 89.
B.A.A. Section, N.S. Wales, 1895–6 — „ „ 457 „
—*B.A.A. Journal*, vol. vi. p. 416, and vol. viii. p. 17.

And, for deeply coloured stars, I have taken about 22 southern objects from the *Red Star Catalogue* of Birmingham-Espin, 1888; also about 30 from Krüger's *Catalog der farbigen Sterne*, Kiel, 1893 (between 0° and −23°), and a few from other sources.

This makes a total of some 2450 southern stars, of which about one-third lie between the equator and 25° south declination; allowing for duplication of objects, there are less than 900 left for the region between −25° and the South pole. Out of the foregoing the most systematic piece of work was that done by the N.S. Wales section of the B.A.A., for they noted the colour of practically every bright southern star up to −25°, to about 4·7 magnitude. Their results, when analysed, yielded a coefficient of mean square contingency with value $C_2 = \cdot 71$, as determined by Dr. Alice Lee (*Monthly Notices*, vol. lxviii. p. 422), which must be considered fairly satisfactory. The duplication of many of these objects by other observers has also been a valuable means of checking the results. At some future time I may give an analysis of *all* these Southern observations, to see how they will compare, when using mean values, with the Northern ones.

Having now summarised what *has* been done in the way of determining the colours of stars in the southern hemisphere, we can briefly indicate what remains to be done in order to complete the survey of the whole heavens on a homogeneous plan. Between the South pole and −25° we have no less than some 2080 stars, from about 4·7 down to 6·5 magnitude, of whose colours we are entirely ignorant. And between 0° and −25° there are nearly 800 more, mostly 6 to 6·5 magnitude, which I have already commenced to observe. Practically, then, we have a total of nearly 2900 stars to be observed before we can arrive at a satisfactory generalisation of the colours of the 9000 or so which are distributed over the entire sky down to 6·5 magnitude. Cannot the enthusiasm of some Southern observers, possessed of adequate optical means and normal colour perception, be evoked to complete this interesting work? It might be very well managed in a twelvemonth, considering the favourable opportunities afforded by southern skies — and the ideal is worth striving after. The classification need only be broad—eschewing minute differences; and following, for the sake of uniformity, the Potsdam scale (with or without the modification I suggested in the "orange" division), and always taking the *mean* of at least three estimates, on different nights. Approximate co-ordinates (or merely the *H.R.* number?) as well as the name of the star should be given, in order to ensure correct identification.

Uxbridge: 1909 *August* 25.

[*Note.*—

Note.—Dunlop's MS., which I have been permitted to examine and make use of, demands a passing notice. The observations are in rough diary form, have no co-ordinates, and there is much repetition. After considerable trouble I discovered the source from whence his star lists were derived, and, seeing his plan of work, was then able to identify nearly all his objects. Instead of being, as I at first expected, a list of merely the bright stars in the Southern hemisphere, it includes many stars down to 5 or 6 magnitude, between 12^h to 18^h and 22^h to 0^h R.A. Had it been completed on this plan, we should have possessed a valuable record of southern star colours at this early date. His estimates are fairly correct (excluding such vague terms as "dusky," "dun white," "tan colour," etc.!) when compared with other observers' work, and checked by the spectrum criterion.

Note on certain Coefficients appearing in the Algebraical Development of the Perturbative Function. (*Second Paper.*) By R. T. A. Innes, Director of the Transvaal Observatory.

In the first paper (*Monthly Notices*, lxix. pp. 633–650) it was pointed out that the quantities $b_{i,3}$, $b_{i,5}$, etc. can be easily and accurately calculated by means of the $b_{i,1}$ and the logarithmic derivatives of these quantities, and that the derivatives follow in a similar simple manner.

Expressions for this purpose are as follows:—

$$\begin{aligned}
&\mathrm{D}^m\times && \mathrm{D}^m\times\\
&\tfrac{1}{2}(b_{i-1,3}+b_{i+1,3}) = && (\mathrm{D}^2+\mathrm{D}-i^2-0)b_{i,1}\\
&\tfrac{1}{2}(b_{i-1,3}-b_{i+1,3}) = && (\qquad\qquad +i)b_{i,1}\\
&\tfrac{1}{2}(b_{i-1,5}+b_{i+1,5}) = \tfrac{1}{9}&&(\mathrm{D}^2+\mathrm{D}-i^2-2)b_{i,3}\\
&\tfrac{1}{2}(b_{i-1,5}-b_{i+1,5}) = \tfrac{1}{9}&&(\qquad\qquad +3i)b_{i,3}
\end{aligned}$$

and generally

$$\begin{aligned}
\tfrac{1}{2}(b_{i-1,s+2}+b_{i+1,s+2}) &= \frac{1}{s^2}\left(\mathrm{D}^2+\mathrm{D}-i^2-\frac{s^2-1}{4}\right)b_{i,s}\\
\tfrac{1}{2}(b_{i-1,s+2}-b_{i+1,s+2}) &= \frac{1}{s^2}\left(\qquad\qquad\qquad +si\right)b_{i,s}
\end{aligned}$$

By repeated substitution, $\mathrm{D}^m b_{i,n}$ can thus be reduced to $b_{p,1}$ and its derivatives. For example, we have the following three expressions for $b_{i,5}$, viz.:—

$$\begin{aligned}
9\mathrm{D}^m b_{i,5} &= \mathrm{D}^m(\mathrm{D}+i+1)(\mathrm{D}-i)(\mathrm{D}+i-1)(\mathrm{D}-i+2)b_{i-2,1}\\
&= \mathrm{D}^m(\mathrm{D}+i+1)(\mathrm{D}-i)(\mathrm{D}-i+1)(\mathrm{D}+i+0)b_{i,1}\\
&= \mathrm{D}^m(\mathrm{D}-i+1)(\mathrm{D}+i)(\mathrm{D}-i-1)(\mathrm{D}+i+2)b_{i+2,1}
\end{aligned}$$

In the development of the perturbative function the general algebraical coefficients are written $a'A_i$, $a'B_i$, etc. (Newcomb, v. p. 27; Tisserand, i. p. 300); confining our attention to $a'A_i$, as the formulæ for the other coefficients are quite similar, we have, following Newcomb,

$$a'A_i = b_{i,1} - \tfrac{1}{2}\sigma^2(b_{i-1,3} + b_{i+1,3}) + \tfrac{3}{8}\sigma^4(b_{i-2,5} + 4b_{i,5} + b_{i+2,5}) - \text{etc.}$$

This transforms to

$$a'A_i = \left[1 - \sigma^2(D^2 + D - i^2) + \frac{\sigma^4}{2^2}\left[(D^2 + D - i^2)^2 - 2(D^2 + D - i^2) + i^2\right] - \text{etc.}\right] b_{i,1}$$

in which only $b_{i,1}$ coefficients appear.

II.

Having, in the first paper, reduced the $b_{i,n+1}$ coefficients to to those of $b_{i,1}$, I gave formulæ for the computation of the $b_{i,1}$ and their derivatives in simple and accurate shapes. The procedure may be outlined as follows:—Having found $b_{0,1}$ by the processes indicated on pp. 638–9, or by means of Mr. Robbins's table on p. 649, the remaining $b_{i,1}$ should be computed by repeated use of the equation

$$b_{i+1,1} = \alpha b_i \times p_{i+1},$$

the quantities p_{i+1} being those referred to on pp. 640–1. The next step is to compute either $D^m b_{0,1}$ or $D^m b_{1,1}$ (p. 637 or p. 642), between which quantities the relation

$$\left.\begin{aligned} D^m b_{0,1} &= \alpha D(D+1)^{m-1} b_{1,1} \\ \text{or} \qquad D(D-1)^m b_{0,1} &= \alpha D^{m+1} b_{1,1} \end{aligned}\right\} \quad . \quad . \quad (A)$$

exists.

One step remains before we can apply the easy equations given on p. 637 to compute the rest of the table: it is to find $Db_{2+p,1}$. For this purpose I offered the equation

$$(D - i)b_{i,1} = (1 + 2i)\beta^2 b_{i,1}(1 - p_{i+1}) \quad . \quad . \quad (B)$$

to which may be added

$$(D + 1)b_{i,1} = (1 - 2i)\beta^2 b_{i,1}(1 - 1/\alpha^2 p_i).$$

But a glance at the formulæ of (A) suggests that they are special cases of a more general expression, and this expectation was speedily realised. The general equation is

$$D^m\left(D + i - \frac{n-1}{2}\right) b_{i,n} = \alpha(D+1)^m\left(D + i + \frac{n-1}{2}\right) b_{i-1,n}.$$

In actual working m may be put $= 0$ and $n = 1$, yielding

$$(D + i)b_{i,1} = \alpha(D + i)b_{i-1,1}$$

and

$$(D - i)b_{i,1} = \alpha(D - i)b_{i+1,1},$$

which equations may advantageously replace (B) in the numerical work. These new equations also supply abundant checks; and obviously they introduce new relations between the hypergeometrical series allied to the $D^m b_{i,n}$.

An easy deduction from the two fundamental equations (A) and (B) is

$$n^2 D^m b_{i,n+2} = a(D+1)^m \left[\left(D + \frac{n+1}{2} \right)^2 - i^2 \right] b_{i,n} .$$

III.

In Radau's "Recherches" on the planetary inequalities of the Moon's motion he introduces the quantities

$$\Delta b_s^i = b_s^i - a b_s^{i-1}$$

in which i may be positive or negative; but the quantities actually used are, as was to be expected, the Radau quantities multiplied by $a^{\frac{s-1}{2}}$.

In the new notation we can compute the completed Radau quantities as follows:—

$$a^{\frac{s-1}{2}} \Delta b_s^i = \frac{1}{n-2} \left(D - i - \frac{n-5}{2} \right) b_{1-i,\, n-2} .$$

[*N.B.*—Radau's s = our n.]

For example,

$$a^2 \nabla b_5^3 = \tfrac{1}{5}(D+3) b_{4,3}$$

and

$$a^3 \Delta b_7^{29} = \tfrac{1}{5}(D-30) b_{28,5} .$$

Using the figures furnished by Brown (*Inequalities in the motion of the Moon due to the direct action of the Planets*, pp. 52–3), the calculations of the last equation compare as follows:—

$b_{29,7} = 1{\cdot}80647$	$\frac{1}{5} D b_{28,5} = 0{\cdot}754990$
$a b_{28,7} = 1{\cdot}67997$	$6 b_{28,5} = 0{\cdot}628494$
$0{\cdot}12650$	$0{\cdot}126496$

Further elaboration is unnecessary, because the powerful D operator can profitably replace the Radau Δ or ∇ operators.

Johannesburg:
1909 *November* 4.

Description of a Field Method for the determination of Latitude with a Theodolite. By N. S. Bartlett.

(*Communicated by W. Ernest Cooke, M.A.*)

In April–May 1906 I took a series of observations for latitude with an 8-inch theodolite on lines suggested to me by Dr. Kurt Hessen, Ph.D., M.A., the Observer attached to the International Latitude Station at Bayswater, near Perth, Western Australia.

The results proved of such an encouraging nature that Mr. H. F. Johnston, the Surveyor-General of this State, decided to procure a supply of high-grade levels, capable of being fitted to 6-inch and 8-inch theodolites. These were obtained through the good offices of Professor Wanach, Central Bureau der Internationalen Erdmessung, and carefully tested by the Bureau before shipment.

Since that date observations have been carried out at a number of Latitude Stations throughout the State, with such uniformly good results that Mr. W. E. Cooke of the Perth Observatory has suggested to me that a short description of the method would be of interest to the Society.

Description.

This method is a combination of the Horrebow-Talcott method with that of circum-meridian altitudes, the times of transit of north and south stars across a fixed horizontal thread being observed at small hour angles east and west of the meridian. These transits are then reduced to the meridian by computation, the difference of zenith distances of north and south stars becomes known, and consequently the latitude.

Selection of Pairs.

By means of a reliable star catalogue a working programme is prepared beforehand. Select two stars, one culminating to the north and another to the south of the observer's zenith, bearing in mind the following restrictions:—

(*a*) Zenith distances to be not less than 15° or more than 50°.

(*b*) Both stars to have approximately the same R.A. and to be about the same magnitude.

(*c*) The difference of zenith distances when on the meridian to be limited to from 15′ to 20′.

Having selected a pair, assume a Z.D. setting from 30″ to 1′ greater than the larger Z.D., and with this datum and the best latitude available compute the hour angle from the meridian and the azimuth of each star. This can be done with three-figure logs.

The approximate times of transit must be known beforehand, so that the four transits may be made to fit comfortably, giving ample time for reading the level, noting times, and setting azimuths.

With a little care a list of 6 to 8 pairs can usually be selected for, say, four hours' work, the results from which should give a good determination of latitude. It is, however, our practice to get three good clear nights at each station.

Observing.

The instrument should be carefully levelled and placed on a firm foundation (the stump of a tree makes an excellent stand). A few minutes before the time anticipated for the first transit set the vertical arc of the theodolite to the assumed zenith distance as closely as possible and firmly clamp. It is of course most essential that the altitude screw be not touched after the first star is transited; but should the instrument get out of level, it can be brought up by the plate screws.

Now set the horizontal circle by hand within a few minutes of the computed azimuth of the first transit, and as the star moves nearly along the horizontal wire observe as closely as possible the exact moment when it is bisected by the wire. The level is now read and the three remaining transits observed in a similar way.

Note.—It is not advisable to clamp the upper plate of the instrument, as it has a disturbing effect on the sensitive level. It is my invariable practice to remove this clamp when observing for latitude, to avoid using it inadvertently.

More reliable results will be attained when all four transits are obtained; but if one should be missed, the value of it can be computed from the star's R.A. This involves a knowledge of the error of the chronometer, and observations for time will be necessary, either before or after the latitude work, or in an interval between two pairs.

Formulæ.

For computing the approximate hour angles and azimuths.

$$\frac{2\sin^2\frac{1}{2}t}{\sin 1''} = \frac{x}{\cos\phi\cos\delta\operatorname{cosec}\text{Z.D.}} \quad . \quad . \quad (1)$$

$$\sin\text{Az} := \frac{\cos\delta\sin t}{\sin\text{Z.D.}} \quad . \quad . \quad . \quad . \quad (2)$$

x in No. 1 = difference in seconds of arc between zenith distance at culmination and the assumed setting of vertical circle.

For computing the Latitude.

$$\phi = \tfrac{1}{2}(\delta + \delta') + \tfrac{1}{2}(Am - A'm') + \tfrac{1}{2}(Bn - B'n') + \tfrac{1}{2}(l - l') \quad . \quad (3)$$

$$m = \frac{2 \sin^2 \frac{t}{2}}{\sin 1''} \text{ for N. Star : } n = \frac{2 \sin^4 \frac{t}{2}}{\sin 1''} \text{ for N. Star}$$

$m' =$ Do. „ S. „ $n' =$ Do. „ S. „

or $\log n = 2 \log m - 5{\cdot}61546$.

$A = \cos \phi \cos \delta \operatorname{cosec} z$

$\log B = 2 \log A + \log \cot z$

$\delta =$ Declination of North Star

$\delta' =$ „ South „

$t =$ Hour angle $= \tfrac{1}{2}$(transit W. − transit E.)

$z =$ Observed Zenith distance

$l =$ Level error for North Star

$l' =$ „ „ South „

$\phi =$ Latitude.

In this formula compute the terms Am and $A'm'$ with five-figure logs; Bn and $B'n'$ with three-figure logs.

If the hour angles are small and Z.D.'s large, the second term can be neglected. The values of m and n can be obtained from vol. ii. *Chauvenet's Astronomy*, or from any circum-meridian altitude tables. An error in t makes the following error in the latitude $d\phi'' = -A \sin t \, dt''$.

Observations.

Latitude deduced from Observations taken on Mount Maxwell Trigonometrical Station.

Pair.	Date 1908.	Observed Lat.	Mean Lat.
	3 Nov.	31° 59′ 20″·2	
	6 „	17 ·9	
No. 1	9 „	19 ·0	31° 59′ 18″·9
ϵ Gruis (3·7)	10 „	19 ·9	
No. 2567 (5·8)*	11 „	18 ·6	
	13 „	18 ·2	
	3 Nov.	31° 59′ 20″·0	
	6 „	17 ·4	
	7 „	17 ·6	
No. 2	8 „	20 ·4	31° 59′ 18″·2
No. 3384 (4·9)*	9 „	16 ·8	
2 Ceti (4·6)	11 „	18 ·0	
	12 „	17 ·9	
	13 „	17 ·8	

* *Catalogue d'Étoiles Brillantes*, J. Bossert, 1906.

OBSERVATIONS—*continued.*

Pair.	Date 1908.	Observed Lat.	Mean Lat.
	3 Nov.	31° 59′ 18″·9	
No. 3	7 ,,	18 ·7	31° 59′ 18″·8
No. 3539 (5·0)*	11 ,,	18 ·9	
,, 2243 (5·2)*	12 ,,	18 ·2	
	13 ,,	19 ·3	
	6 Nov.	31° 59′ 19″·5	
	7 ,,	17 ·2	
No. 4	8 ,,	19 ·8	
No. 3474 (4·3)*	10 ,,	19 ·6	31° 59′ 19″·2
No. 2386 (5·1)*	11 ,,	19 ·4	
	12 ,,	19 ·4	
	13 ,,	19 ·2	
	6 Nov.	31° 59′ 16″·7	
No. 5	10 ,,	16 ·9	
No. 3371 (4·1)*	11 ,,	16 ·9	31° 59′ 17″·2
No. 2714 (6·0)*	12 ,,	18 ·5	
	13 ,,	16 ·9	
	6 Nov.	31° 59′ 17″·1	
No. 6	10 ,,	18 ·3	
δ Hyd. (4·2)	11 ,,	17 ·4	31° 59′ 18″·0
υ Ceti (5·0)	12 ,,	19 ·5	
	13 ,,	17 ·6	
		Adopted Latitude	31° 59′ 18″·5

These observations were taken by me in November of last year with an 8-inch transit theodolite by Simms, fitted with a level, one division of which had a value of 1″. The observations were taken close to the Mt. Maxwell Trigonometrical Station, naturally a very exposed spot; there was no protection for the instrument, which was mounted on a large and very solid red-gum stump. The weather was for the most part unfavourable, and in many instances transits were lost on account of clouds, and had to be computed. The times were taken by means of a small pocket chronometer rated to sidereal time, and a stop-watch.

In order to show the small amount of computing involved in deducing the latitude, I give the computation in connection with Pair No. 3 on the night of 12th November.

* *Catalogue d'Étoiles Brillantes*, J. Bossert, 1906.

EXAMPLE.

North.

No. 2243 (5·2)*

	h	m	s	
Obsd.	0	44	31·7	W.
,,	0	17	51·7	E.
	2)	26	40·0	
	13	20·0	$-t$	

$\cos\phi = 9{\cdot}92847$
$\cos\delta = 9{\cdot}99889$
$\operatorname{cosec} z = 0{\cdot}32743$
$A = 0{\cdot}25479$
$\log m = 2{\cdot}54279$
$2{\cdot}79758 = 627''{\cdot}46 = Am$

$2 \log A = 0{\cdot}509$
$\cot z = 0{\cdot}273$
$\log n = 9{\cdot}470$
$0{\cdot}252 = 1''{\cdot}79 = Bn$

Level.

Eye.	*Object.*
8·6	11·1
9·2	10·4
−17·8	+21·5
	−17·8

$+ \frac{3{\cdot}7}{4} \times 1'' = +0''{\cdot}92 = l$

South.

No. 3539 (5·0)*

	h	m	s	
Obsd.	0	49	57·2	W.
,,	0	23	43·2	E.
	2)	26	14·0	
	13	7·0	$-t'$	

$\cos\phi = 9{\cdot}92847$
$\cos\delta' = 9{\cdot}69935$
$\operatorname{cosec} z' = 0{\cdot}32743$
$A' = 9{\cdot}95525$
$\log m' = 2{\cdot}52856$
$2{\cdot}48381 = 304''{\cdot}66 = A'm'$

$2 \log A' = 9{\cdot}910$
$\cot z' = 0{\cdot}273$
$\log n' = 9{\cdot}440$
$9{\cdot}623 = 0''{\cdot}42 = B'n'$

* *Catalogue d'Étoiles Brillantes*, J. Bossert, 1906.

Level.

		Eye.	*Object.*
$Am =$	$627''{\cdot}46$	$6{\cdot}4$	$13{\cdot}2$
$A'm' =$	$-304\ {\cdot}66$	$10{\cdot}5$	$9{\cdot}0$
	$2)+322''{\cdot}80$	$-16{\cdot}9$	$+22{\cdot}2$
	$+161''{\cdot}40$		$-16{\cdot}9$
			$+\frac{5{\cdot}3}{4}\times 1'' = +1''{\cdot}32 = l'$

$Bn = +1''{\cdot}79$
$B'n' = -0\ {\cdot}42$

$2)+1''{\cdot}37$
$+0''{\cdot}68$

$\frac{1}{2}(\delta+\delta') = -32^\circ\ 1'\ 59''{\cdot}52$
$\frac{1}{2}(Am - A'm') = +\quad 2\ 41\ {\cdot}40$
$\frac{1}{2}(Bn - B'n') = +\quad 0\ {\cdot}68$
$\frac{1}{2}(1 - 1') = -\quad 0\ {\cdot}20$

$l = +0''{\cdot}92$
$l' = -1\ {\cdot}32$

$2)-0''{\cdot}40$
$-0''{\cdot}20$

$\phi = -31^\circ\ 59'\ 17''{\cdot}64$
$0\ {\cdot}58$ reduction to Stn.

$\phi = -31^\circ\ 59'\ 18''{\cdot}22$ △ Mt. Maxwell

Advantages of Method.

(*a*) No readings of the circle are taken, the instrument being set at the same altitude for both north and south stars; the observations therefore are quite free from errors of graduation.

(*b*) Errors of refraction are practically eliminated, as stars are transited at same altitude north and south. With a good pair, one complete determination of latitude is made in about thirty minutes, so that any change in refraction is reduced to a minimum.

(*c*) Errors of adjustment of the instrument have no effect on the observations; the only precaution necessary is to ensure the horizontality of the horizontal thread.

(*d*) As a micrometer eyepiece is not required, the trouble and errors induced in this delicate mechanism by dust, heat, and transport are entirely avoided.

In a vast unsettled country like Western Australia this is an important consideration, and is one which, I think, should commend the method to surveyors working in places where all the conveniences of civilisation are conspicuous by their absence.

It is not claimed that there is anything entirely new and original in this method (Chandler and Chauvenet both give it a place in their works), but its adaptation to observations with a theodolite and actual *field* operations is, I think, a new departure.

Surveyor-General's Office,
Perth, Western Australia:
1909 *October* 7.